HINTS AND SOLUTIONS OF THE EXAMPLES

IN 'ANALYTICAL PLANE GEOMETRY'

FOR

PRE-ENGINEERING & FIRST YEAR OF
THREE-YEAR DEGREE COURSE

BY

BANSI LAL, M.A.

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PREFACE

The recently raised standard of Examinations in Mathematics of Pre-Engineering and First Year of Three-Year Degree Course and the new style of Examination papers require ingenuity and skill on the part of the Examinees to try questions. Hence Hints and Solutions of the Examples in my *Analytical Plane Geometry* have been given **separately** in this booklet and not in the Text-book itself. This arrangement, it is hoped, will encourage independent effort and minimize the chances of temptation to the students to see the Hints or the Solutions without giving a fair trial to the Examples.

In preparing these *Hints and Solutions* two objects have been kept in view :

(1) to save the time and lighten the work of the teacher who is already burdened on account of the extra-heavy Syllabus in Mathematics and the limited time at his disposal, and

(2) to set private students on the right path.

Purposely I have not written a "complete key" so that there may be some demand on the thought and intelligence of the student to try at least the Examples which follow immediately from the Article or which are exactly of the same type as those worked out in the Text-book or as those for which Hints are given in this booklet. For, I firmly believe that books like *"Analytical Plane Geometry (Solved)"* will not help the student in the Examination hall unless he has done some Examples with his own unaided effort at home.

D. A. V. COLLEGE,
JULLUNDUR CITY.
1st July, 1965.

BANSI LAL

HINTS AND SOLUTIONS

CHAPTER I

Pages 5—6.

3. $a \cos \beta - a \cos \alpha = a(\cos \beta - \cos \alpha) = 2a \sin \frac{\beta+\alpha}{2} \sin \frac{\alpha-\beta}{2}$,

$a \sin \beta - a \sin \alpha = a(\sin \beta - \sin \alpha) = 2a \cos \frac{\beta+\alpha}{2} \sin \frac{\beta-\alpha}{2}$

$$= -2a \cos \frac{\beta+\alpha}{2} \sin \frac{\alpha-\beta}{2}.$$

5. Let the ordinate $= y$.

7. Draw a rough Fig. Let A, B, C, D be the pts. $(0, -1)$, $(2, 1)$, $(0, 3)$, $(-2, 1)$.

(*i*) Show that AB$=$BC$=$CD$=$DA, $\therefore$ ABCD is a rhombus ...(1)

(*ii*) Show that $AC^2 = AB^2 + BC^2$, $\therefore$ $\angle$ABC$=$1 rt. $\angle$ ...(2)

$\therefore$ from (1) and (2), ABCD is a square.

8. Draw a rough Fig. Let A, B, C be the pts. $(4, 3)$, $(-2, 3)$, $(6, -1)$, and S the pt. $(1, -1)$ within the $\triangle$ABC.

Show that SA$=$SB$=$SC.

Page 10.

1. (*a*) "Divides in a given ratio" means "divides in a given ratio (*i*) *internally*, and (*ii*) *externally*. So reproduce both Arts. 6 (*a*) and 6 (*b*).

2. (*a*) Proceed as in Art. 6 (*a*).

3. Proceed as in solved Ex. 3, Art. 6 (*b*).

4. Draw a rough Fig. Let A, B, C, D be the pts. $(1, -1)$, $(-2, 2)$, $(4, 8)$, $(7, 5)$.

Test for a rectangle. (*i*) Show that ABCD is a $\|$gm.
(*ii*) Show that $\angle$ABC$=$1 rt. $\angle$.

Page 12.

2. Let A, B, C be the pts. (x_1, y_1), (x_2, y_2), (x_3, y_3). Then D, the mid-pt. of BC, is $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$.

$\therefore$ the co-ordinates of the *centroid* G *which divides AD internally in the ratio 2 : 1*, are $x = ?$, $y = ?$

3. Proceed as in solved Ex., Art. 7 (*b*).

Page 15.

1. Take D as the origin and DC as the x-axis, so that DY, the line thro' D$\perp$ to DC, is the y-axis. Let BC$=2a$. Then D is $(0, 0)$, C is $(a, 0)$, B is $(-a, 0)$, and let A be (x_1, y_1).

Then $AB^2+AC^2=?$

$2(AD^2+DC^2)=?$

Page 19.

2. (*b*) By the Rule of Note 1, Art. 11, it will be found that

$$\triangle=\tfrac{1}{2}ab\,[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\gamma)]$$
$$=\tfrac{1}{2}ab\left[2\sin\frac{\gamma-\alpha}{2}\cos\frac{2\beta-\alpha-\gamma}{2}-2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2}\right]$$
$$\left[\because \sin(\alpha-\gamma)=\sin[-(\gamma-\alpha)]=-\sin(\gamma-\alpha)=-2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2}\right]$$
$$=ab\sin\frac{\gamma-\alpha}{2}\left[\cos\frac{2\beta-\alpha-\gamma}{2}-\cos\frac{\gamma-\alpha}{2}\right]$$
$$=ab\sin\frac{\gamma-\alpha}{2}\cdot 2\sin\frac{\beta-\alpha}{2}\sin\frac{\gamma-\beta}{2}.$$

4. Let P be the pt. (x, y). Then PA$=$PB, and $\triangle$PAB$=10$. Use the complete area formula.

Page 20. Art. 12.

Ex. (*ii*) It will be found that

$$\triangle=-\tfrac{1}{2}a^2[t_1t_2(t_1-t_2)+t_2t_3(t_2-t_3)+t_3t_1(t_3-t_1)]$$
$$=-\tfrac{1}{2}a^2[-(t_1-t_2)(t_2-t_3)(t_3-t_1)]$$
$$[\because \Sigma ab(a-b)=-(a-b)(b-c)(c-a)]$$

Page 20. Art. 13.

3. The three pts. (x, y), $(a, 0)$, $(0, b)$ are collinear.

Page 25.

Ex. "Lay down the position of the point" means "plot the point."

Page 27. Art. 21 (*b*).

1. (*i*) The polar co-ordinates are $\left(1, \frac{\pi}{2}\right)$.

[Compare with (r, θ)]

Here $r=1$, $\theta=\frac{\pi}{2}$

$\therefore$ the rectangular co-ordinates are

$x=r\cos\theta$ [Art. 20 (*a*)]

$=1\cdot\cos\frac{\pi}{2}=0,$

$y=r\sin\theta$ [Art. 20 (*a*)]

$=1\cdot\sin\frac{\pi}{2}=1,$

i.e. $(0, 1)$.

2. (*i*) Use the Rule of Art. 21 (*a*).

4. (*i*) Use the Rule of Art. 21 (*b*).

(*ii*) The equation is $\theta=\tan^{-1}m$,

or $\tan\theta=m$ or $\dfrac{\sin\theta}{\cos\theta}=m$.

Use the Rule of Art. 21 (*b*).

6. (*ii*) The equation is $r^{\frac{1}{2}}\cos\frac{\theta}{2}=a^{\frac{1}{2}}$

or, squaring, $r\cos^2\frac{\theta}{2}=a$ or $\dfrac{r(1+\cos\theta)}{2}=a$

[Use the Rule of Art. 21 (*b*)]

$\left[\text{Put }\cos\theta=\frac{x}{r}\right]$

or $r\left(1+\frac{x}{r}\right)=2a$ or $r+x=2a$ [Put $r=\sqrt{x^2+y^2}$]

or $\sqrt{x^2+y^2}+x=2a$ or, transposing, $\sqrt{x^2+y^2}=2a-x$.

Square.

MISCELLANEOUS EXAMPLES ON CHAPTER I

Pages 27—29.

3. Draw a rough Fig.

Let A, B, C be the pts. $(0, 0)$, $(3, \sqrt{3})$, (x, y).

Then BC$=$CA$=$AB.

$\therefore \sqrt{(x-3)^2+(y-\sqrt{3})^2}=\sqrt{(0-x)^2+(0-y)^2}=\sqrt{(3-0)^2+(\sqrt{3}-0)^2}$

or, squaring,

$x^2-6x+9+y^2-2\sqrt{3}\,y+3=x^2+y^2=12.$

From the first and second members, $-6x-2\sqrt{3}\,y+12=0$

or $3x+\sqrt{3}\,y-6=0$ ...(1)

From the second and third members, $x^2+y^2-12=0$ ...(2)

[To solve (1) and (2) for x and y.]

Substitute the value of $y(=-\sqrt{3}\,x+2\sqrt{3})$ from (1) in (2), and solve the resulting quadratic in x.

Substitute these values of x one by one in $y=-\sqrt{3}\,x+2\sqrt{3}$, and find the corresponding values of y.

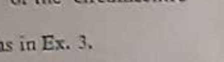
8. (*ii*) Draw a rough Fig. Let A, B, C be the pts. $(2, 1)$, $(5, 2)$,

$(3, 4)$. Let (x, y) be the required co-ordinates of the circumcentre S inside the $\triangle$ABC.

Then SA=SB=SC. ∴ ? Square and solve as in Ex. 3.

9. Let the required ratio be $k : 1$.

The pt. which divides the join of $(7, 9)$ and $(-1, 1)$ in the ratio $k : 1$ is



$$\left[\frac{k(-1)+1(7)}{k+1}, \frac{k(1)+1(9)}{k+1}\right],$$

i.e. $\left(\frac{-k+7}{k+1}, \frac{k+9}{k+1}\right)$ ...(1)

∵ this is the same as $(2, 4)$...(2)

∴ comparing (1) and (2), $\frac{-k+7}{k+1}=2$...(3), $\frac{k+9}{k+1}=4$...(4)

Find the value of k from (3) or (4).

10. Let the required co-ordinates of the other end be (x, y).

The mid-pt. of the join of $(-6, 4)$ and (x, y) is ?...(1)

∵ this is the same as $(-4, -2)$...(2)

∴ compare (1) and (2).

11. Draw a rough Fig. It will be found that PQ=5,

∴ QR=PR−PQ=10−5=5,

∴ PQ=QR, *i.e.* Q is the mid-pt. of PR. Proceed as in Ex. 10.

13. (*a*) Let ABC be the triangle. Choose the axes as in Cor., Art. 10 (*b*). Then C is $(a, 0)$, B is $(-a, 0)$, and let A be (x_1, y_1).

(*b*) Let ABC be the triangle, and D, E, F the mid-pts. of the sides BC, CA, AB.

Take D as the origin and DC as the x-axis, so that DY, the line thro' D ⊥ to DC, is the y-axis.

Let BC=$2a$. Then D is $(0, 0)$, C is $(a, 0)$, B is $(-a, 0)$, and let A be (x_1, y_1).

17. Let A, B, C be the pts. $(3, -4)$, $(-2, 3)$, $(-6, 5)$.

Let D, the fourth vertex of the ∥gm., be (x, y).

The mid-pt. of AC is ?

The mid-pt. of BD is ?

∵ ABCD is a ∥gm.

∴ the mid-pt. of AC is the same as the mid-pt. of BD.

Find x and y.

Now area of the ∥gm. ABCD=2$\triangle$ABC.

18. (*b*) The equation is $r^{\frac{1}{2}} = a^{\frac{1}{2}} \sin\frac{\theta}{2}$

or, squaring, $r=a\sin^2\frac{\theta}{2}=\frac{a(1-\cos\theta)}{2}$

[Use the Rule of Art. 21 (*b*)]

$\left[\text{Put } \cos\theta=\frac{x}{r}\right]$

or $r=\frac{a}{2}\left(1-\frac{x}{r}\right)=\frac{a}{2r}(r-x)$ or $2r^2=a(r-x)$

[Put $r=\sqrt{x^2+y^2}$]

or $2(x^2+y^2)=a(\sqrt{x^2+y^2}-x)$ or, transposing,

$2(x^2+y^2)+ax=a\sqrt{x^2+y^2}$.

Square.

19. (*i*) The equation is $r(\cos 3\theta+\sin 3\theta)=5k\sin\theta\cos\theta$

or $r(4\cos^3\theta-3\cos\theta+3\sin\theta-4\sin^3\theta)=5k\sin\theta\cos\theta$

[Use the Rule of Art. 21 (*b*)]

$\left[\text{Put } \cos\theta=\frac{x}{r}, \sin\theta=\frac{y}{r}\right]$

or $r\left[4\frac{x^3}{r^3}-3\frac{x}{r}+3\frac{y}{r}-4\frac{y^3}{r^3}\right]=5k\frac{y}{r}\cdot\frac{x}{r}$

or $4x^3-3xr^2+3yr^2-4y^3=5kxy$.

[Put $r=\sqrt{x^2+y^2}$ or $r^2=x^2+y^2$]

CHAPTER II

Page 32.

2. (*i*) Let (x, y) be *any* pt. on the locus.

(*ii*) Then distance of (x, y) from $(1, -2)$

=distance of (x, y) from $(-3, -4)$.

(*iii*) ∴ $\sqrt{(x-1)^2+(y+2)^2}=\sqrt{(x+3)^2+(y+4)^2}$.

Square and simplify.

4. $1-k^2$ cancels thro' out.

5. (*i*) Let (x, y) be *any* pt. on the locus.

(*ii*) Then square of distance of (x, y) from $(a, 0)$

−square of distance of (x, y) from $(-a, 0)=2k^2$...(1)

or square of distance of (x, y) from $(-a, 0)$

−square of distance of (x, y) from $(a, 0)=2k^2$...(2)

(*iii*) It will be found from (1) that the equation of the locus is

$2ax+k^2=0$ ...(3)

and from (2), the equation is $2ax-k^2=0$...(4)

∴ combining (3) and (4), the equation of the locus is $2ax\pm k^2=0$.

9. Proceed as in solved Ex. 3, Art. 24.

10. Use the complete area formula.

CHAPTER III

Page 35.

2. (*i*) Let the required equation of the line be $x=a$ ...(1)

[Art. 28]

Proceed as in solved Ex. 2, Art. 28.

Page 36.

1. (*ii*) The line is the y-axis.

Pages 40—41.

4. Let the intercepts be $a, -a$.

5. The intercepts are
 (*i*) equal.
 (*ii*) equal in magnitude but opposite in sign.

7. (*i*) Let the required equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(1)$$

(*ii*) The pt. which divides the join of $(a, 0)$ and $(0, b)$ in the ratio $1 : 2$ is $\left(\frac{2a}{3}, \frac{b}{3}\right)$...(2)

$\because$ this is the same as $(-4, 1)$...(3)

$\therefore$ compare (2) and (3).

Page 44.

1. The equation is $lx+my-1=0$...(1)

[**Method of comparison**]

Case I. If $m \neq 0$, [To solve (1) for y]

then from (1), transposing,

$$my=-lx+1.$$

Dividing thro' out by m, (which $\neq 0$),

$$y=-\frac{l}{m}x+\frac{1}{m},$$

which is of the form "$y=mx+c$",

and $\therefore$ represents a st. line.

Case II. If $m=0$, then (1) becomes

$lx-1=0$. [To solve for x]

Transposing, $lx=1$.

Dividing by l, (which $\neq 0$),

[$\because$ if $l=0$, then m being already $=0$, (1) becomes $-1=0$, which is impossible]

$$x=\frac{1}{l},$$

which is of the form $x=a$,

and $\therefore$ represents a st. line.

3. Choose the axes as in Art. 10 (*a*). It will be found that the equation of the locus is $x+y=a$, which, being an equation of the first degree in x and y, represents a st. line (Art. 36).

4. Choose the axes as in Art. 10 (*b*).

6. (*i*) The equation of the x-axis is $y=0$;
 (*ii*) the equation of the y-axis is $x=0$.

7. The equation of the line is $y-x+2=0$ ...(1)

The pt. which bisects the join of $(2, 1)$ and $(-2, -5)$ is

$$\left[\frac{2+(-2)}{2}, \frac{1+(-5)}{2}\right], \text{ i.e. } (0, -2).$$

Substituting the co-ordinates of this pt. in (1), we get

$-2-0+2=0$ or $0=0$, which is true.

$\therefore$ the line (1) bisects the join of $(2, 1)$ and $(-2, -5)$.

Page 46.

2. (i) Use the Rule of Cor., Art. 38.

Pages 48—49.

2. (*iii*) The equation is $x-y=0$.

(i) Dividing thro' out by $\sqrt{(1)^2+(-1)^2}$

[$\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2}$]

i.e. by $\sqrt{2}$, $\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y=0$.

(ii) Here the constant term is 0, and coeff. of $y=-\frac{1}{\sqrt{2}}$, which is $-$ve

$\therefore$ changing the signs thro' out

[*To make coeff. of y on L.H.S.* +*ve* (Art. 39, Note 3)]

$$-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y=0 \quad ...(1)$$

Comparing this with $x \cos \alpha+y \sin \alpha=p$,

$$\cos \alpha=-\frac{1}{\sqrt{2}}, \ \sin \alpha=\frac{1}{\sqrt{2}}, \ p=0.$$

Proceed as in solved Ex. (*b*), Art. 39.

Pages 51—52.

2. Let A, B, C be the pts. $(1, 4)$, $(2, -3)$, $(-1, -2)$. Find the equations of the sides BC, CA, AB.

4. (*ii*) It will be found that the equation of the median is $y=\frac{2}{0}(x-1)$. Cross-multiplying, $y(0)=2(x-1)$ or $x-1=0$.

Note. Important. *If in the two-pt. form, we get zero in the denominator, we cross-multiply.*

9. Find the equation of the line passing thro' the second pair of pts., namely thro' (3, 2) and (5, 0), and proceed as in solved Ex., Art. 36.

10. Let the line meet the axes in A, B. Find the intercepts OA, OB on the axes. Then $AB=\sqrt{OA^2+OB^2}$.

Page 54.

2. The equation of the line, in the distance form, is

$$\frac{x-1}{\cos 60^\circ}=\frac{y-\sqrt{3}}{\sin 60^\circ}=r$$

or

$$\frac{x-1}{\frac{1}{2}}=\frac{y-\sqrt{3}}{\frac{\sqrt{3}}{2}}=r.$$

$\therefore$ *any* pt. on the line is $\left(1+\frac{r}{2}, \sqrt{3}+\frac{\sqrt{3}\,r}{2}\right)$.

If it lies on the line $x+\sqrt{3}\,y=8$, then ?

Find the value of r.

4. (*i*) $r=\sqrt{2}$; (*ii*) $r=-\sqrt{2}$.

5. Let θ be the required inclination of the line to the x-axis. Then the equation of the line, in the distance form, is

$$\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=r.$$

Any pt. on the line is $(1+r\cos\theta, 2+r\sin\theta)$. Here $r=\frac{1}{3}\sqrt{6}$.

$\therefore$ the pt. is $\left(1+\frac{1}{3}\sqrt{6}\cos\theta, 2+\frac{1}{3}\sqrt{6}\sin\theta\right)$.

$\because$ it lies on the line $x+y=4$

$\therefore 1+\frac{1}{3}\sqrt{6}\cos\theta+2+\frac{1}{3}\sqrt{6}\sin\theta=4$ or $\frac{\sqrt{6}}{3}(\sin\theta+\cos\theta)=1$

or $\quad \sin\theta+\cos\theta=\frac{3}{\sqrt{6}}$.

Dividing thro' out by $\sqrt{(1)^2+(1)^2}$*, *i.e.* by $\sqrt{2}$,

$$\sin\theta.\frac{1}{\sqrt{2}}+\cos\theta.\frac{1}{\sqrt{2}}=\frac{3}{\sqrt{12}}=\frac{3}{2\sqrt{3}}$$

or $\quad \sin\theta\cos 45^\circ+\cos\theta\sin 45^\circ=\frac{\sqrt{3}}{2}$

or $\quad \sin(\theta+45^\circ)=\sin 60^\circ$ or $\sin(180^\circ-60^\circ)$

$\quad =\sin 60^\circ$ or $\sin 120^\circ$

*See the Rule in Note 2, Art. 92 in the author's *New Elementary Plane Trigonometry* (11th Edition).

$\therefore \quad \theta+45^\circ=60^\circ$ or 120°

or $\quad \theta=15^\circ$ or 75°.

Page 55.

Ex. Comparing coeffs., $\frac{A}{\cos\alpha}=\frac{B}{\sin\alpha}=\frac{C}{p}$,

$\therefore \cos\alpha=\frac{A}{C}p$...(1), $\sin\alpha=\frac{B}{C}p$...(2)

Eliminate α from (1) and (2) [by squaring (1) and (2), and adding].

Page 59.

3. It will be found that $\tan\theta=-3$...(1)

If ϕ is the acute angle between the lines, then changing the sign in (1) [Art. 45 (*a*), Note 2],

$\tan\phi=3$, $\therefore \phi=\tan^{-1}3$.

5. "Find the angle between the two lines" means, for definiteness, "find the *acute* angle between the two lines." See Art. 45 (*a*), Note 2.

Page 60.

Ex. Let A, B, C, D be the pts. (1, −2), (2, 0), (1, 6), (0, 4).

Then m_1=slope of AC$=\frac{6-(-2)}{1-1}=\frac{8}{0}$, $\therefore m_1$ is infinite

$\therefore$ *the angle formula* $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ *fails.*

[See Art. 45 (*b*), Note]

It will be found that the equation of AC is $x-1=0$ ...(1)

[Form : $Ax+By+C=0$]

and that of BD is $2x+y-4=0$ ...(2)

[Compare (1) with $a_1x+b_1y+c_1=0$,

and (2) with $a_2x+b_2y+c_2=0$]

Here $a_1=1$, $b_1=0$; $a_2=2$, $b_2=1$.

$\therefore$ if θ is the angle between the diagonals,

then $\tan\theta=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}$ [Angle formula (Second form)]

$$=\frac{2(0)-1(1)}{1(2)+0(1)}=-\tfrac{1}{2}.$$

Page 61.

1. Choose the axes as in Cor., Art. 10 (*b*).

Pages 63—64.

2. Here m_1 = slope of the line passing thro' (1, 2) and (2, −2)

$$= \frac{-2-2}{2-1} \qquad \left[\frac{y_2-y_1}{x_2-x_1}\right]$$
$$= -4,$$

m_2 = slope of the line passing thro' (8, 2) and (4, 1)

$$= \frac{1-2}{4-8} = \frac{-1}{-4} = \frac{1}{4}.$$

3. Let A, B be the pts. (1, 1), (2, 3). Then D, the mid.-pt. of AB, is $\left(\frac{1+2}{2}, \frac{1+3}{2}\right)$, *i.e.* $\left(\frac{3}{2}, 2\right)$. Find the equation of the line thro' D ⊥ to AB. [Proceed as in solved Ex., Art. 48 (*a*).]

Page 66.

3. (*i*) First write the equation of the given line in the general form :

$$2x-3y+1=0. \qquad [\text{ Form : } Ax+By+C=0 \ ]$$

4. (*i*) The equation of *any* line ⊥ to $lx+my-1=0$ is

$$mx-ly+p=0 \ldots(1)$$

Note. Why to use p (and not k). In order to avoid confusion with the given k in (h, k) we have taken the equation of *any* line ⊥ to $lx+my-1=0$ as $mx-ly+p=0$ (and *not* $mx-ly+k=0$).

Pages 69—70.

2. The foot of the ⊥ is the pt. of intersection of the line $y=3x+4$ and the line thro' (2, 3) ⊥ to $y=3x+4$.

4. Let A, B, C be the pts. (1, 0), (2, −4), (−5, −2). Find the equation of the line thro' A ⊥ to BC, and then the equation of the line thro' B ⊥ to CA. Find the pt. of intersection of these two lines.

7. Draw a rough Fig. The equations of the sides of the triangle are

$$y=0 \ldots(1), \quad x\cos\alpha+y\sin\alpha-p=0 \ldots(2), \quad x=0 \ldots(3)$$

Solving (2) and (3), then (3) and (1), and then (1) and (2), the vertices are $A\left(0, \frac{p}{\sin\alpha}\right)$, B (0, 0), $C\left(\frac{p}{\cos\alpha}, 0\right)$.

∴ *the lengths of the sides are*

$$a = BC = \frac{p}{\cos\alpha},$$

$$b = CA = \sqrt{\frac{p^2}{\cos^2\alpha}+\frac{p^2}{\sin^2\alpha}} = \sqrt{\frac{p^2(\sin^2\alpha+\cos^2\alpha)}{\cos^2\alpha\sin^2\alpha}} = \frac{p}{\cos\alpha\sin\alpha},$$

$$c = AB = \frac{p}{\sin\alpha}.$$

Now use the incentre formulae :

$$x = \frac{ax_1+bx_2+cx_3}{a+b+c}, \; y = \frac{ay_1+by_2+cy_3}{a+b+c}. \qquad [\text{ Art. 8 } (b) \ ]$$

Page 72.

3. Solve the first and third equations (free from the unknown quantity a), and substitute these values of x and y in the second equation.

Page 76.

6. *If a line cuts off equal intercepts a, a from the axes, its equation is* $\frac{x}{a}+\frac{y}{a}=1$ *or* $x+y=a$, ∴ **its slope = −1.**

7. If a line is equally inclined to the axes, *its slope* = ±1. [Art. 32, Ex. 3]

Pages 77—78.

1. Proceed as in Art. 57 (*a*).

Pages 81—82.

1. (*a*) Proceed as in Art. 57 (*b*).

7. (*i*) Proceed as in solved Ex. 2, Art. 57 (*b*).

10. Let the required pt. be $(x, 0)$. Then the ⊥ distance of $(x, 0)$ from the line $\frac{x}{a}+\frac{y}{b}=1$ is a. Use the *complete* ⊥ distance formula [Note 1 after Cor., Art. 57 (*b*)].

Note. *When the length of the ⊥ is given, use the complete ⊥ distance formula.* [Note 3 after Cor., Art. 57 (*b*)]

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3. Find the vertices of the triangle, and show that each vertex and the origin are on the same side of the opposite side of the triangle.

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2. (*ii*) It will be found that the equations of the bisectors are

$$x\cos\alpha+y\sin\alpha-p=\pm(x\cos\beta+y\sin\beta-p'),$$

i.e. $\quad x(\cos\alpha-\cos\beta)+y(\sin\alpha-\sin\beta)-(p-p')=0 \ldots(1)$

and $\quad x(\cos\alpha+\cos\beta)+y(\sin\alpha+\sin\beta)-(p+p')=0 \ldots(2)$

From (1), $m_1 = -\frac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta} \qquad \left[-\frac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 37, Cor.)}\right]$

from (2), $\quad m_2 = -\frac{\cos\alpha+\cos\beta}{\sin\alpha+\sin\beta}.$

$$\therefore \quad m_1 m_2 = \frac{\cos^2 \alpha - \cos^2 \beta}{\sin^2 \alpha - \sin^2 \beta} = \frac{(1-\sin^2 \alpha)-(1-\sin^2 \beta)}{\sin^2 \alpha - \sin^2 \beta}$$

$$= \frac{-(\sin^2 \alpha - \sin^2 \beta)}{\sin^2 \alpha - \sin^2 \beta} = -1,$$

$\therefore$ the bisectors are $\perp$.

MISCELLANEOUS EXAMPLES ON CHAPTER III

Pages 89—92.

2. Let OA $= a$, OB $= b$, then $a+b=8$...(1)

Take OA, OB as the axes. Then A is $(a, 0)$, B is $(0, b)$.

$\therefore$ the co-ordinates of the mid-pt. of AB are

$$x = \frac{a}{2}, \; y = \frac{b}{2} \quad \text{...(2)}$$

Eliminate a, b from (1) and (2) [by substituting the values of a, b from (2) in (1)].

3. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$.

Then $\because$ it passes thro' (h, k)

$$\therefore \frac{h}{a} + \frac{k}{b} = 1 \quad \text{...(1)}$$

The co-ordinates of R are

$$x = a, \; y = b \quad \text{...(2)}$$

Eliminate a, b from (1) and (2) [by substituting the values of a, b from (2) in (1)].

6. Take the Fig. of Art. 34.

$\angle$OBL $= 90° - \angle$LOB $= \angle$AOL $= \alpha$.

$a =$ OA $\left[\text{But } \frac{\text{OA}}{\text{OL}} = \sec \alpha, \; \therefore \text{ OA} = \text{OL} \sec \alpha \right]$

$= p \sec \alpha,$

$b =$ OB

$\left[\text{But } \frac{\text{OB}}{\text{OL}} = \text{cosec OBL} = \text{cosec } \alpha, \; \therefore \text{ OB} = \text{OL cosec } \alpha \right]$

$= p \text{ cosec } \alpha.$

Substitute these values of a, b in the equation of the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

nd multiply the resulting equation by p.

9. The equation of the line, in the distance form, is

$$\frac{x-x_1}{\cos \theta} = \frac{y-y_1}{\sin \theta} = r.$$

Any pt. on the line is $(x_1 + r \cos \theta, y_1 + r \sin \theta)$. If it lies on the line $Ax + By + C = 0$, then

$A(x_1 + r \cos \theta) + B(y_1 + r \sin \theta) + C = 0$, find the value of r.

10. The equation of the given line is

$$\frac{x}{\cos \theta} + \frac{y}{\sin \theta} = a \text{ or } x \sin \theta + y \cos \theta - a \sin \theta \cos \theta = 0.$$

11. The equations of the lines are $x = 1$, $x + y = 4$

or $\quad x - 1 = 0 \quad$ ...(1)

$x + y - 4 = 0 \quad$ ...(2)

It will be found that the equation of the line thro' the pt. of intersection of the lines (1) and (2) and $\perp$ to the line $x = 2y$ is

$2x + y - 5 = 0 \quad$ ...(3)

To find the angle between the lines (1) and (3).

Compare (1) and (3) with

$$a_1 x + b_1 y + c_1 = 0,$$

and
$$a_2 x + b_2 y + c_2 = 0,$$

and use the angle formula (Second form)

$$\tan \theta = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}. \quad \text{[Art. 45 (b), Note]}$$

12. The slope of the line joining the pts. $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$

$$= \frac{a \sin \beta - a \sin \alpha}{a \cos \beta - a \cos \alpha} = \frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha}$$

[Use the "C, D" formulae of Elementary Trigonometry]

$$= -\cot \frac{\alpha + \beta}{2}.$$

It will be found that the equation of the line thro' the origin $\perp$ to the above line is

$$y = x \tan \frac{\alpha + \beta}{2} \quad \text{...(1)}$$

The co-ordinates of the mid-pt. of the join of $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$ are

$$x = \frac{a \cos \alpha + a \cos \beta}{2} = a \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2},$$

$$y = \frac{a \sin \alpha + a \sin \beta}{2} = a \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.$$

Substitute the co-ordinates of this pt. in (1), and show that the equation is satisfied.

14. Draw a rough Fig. Two sides pass thro' (1, 2), and make

an angle of 45° with the diagonal passing thro' (1, 2) and (3, 8).

The other two sides pass thro' (3, 8), and make an angle of 45° with the above diagonal.

The second diagonal passes thro' the mid-pt. of the join of (1, 2) and (3, 8), and is ⊥ to this join.

15. (*ii*) Solve the two equations by cross-multiplication, and use

$$\sin\theta_2-\sin\theta_1=2\cos\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2},$$

$$\cos\theta_1-\cos\theta_2=2\sin\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_2-\theta_1}{2},$$

$$\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1=\sin(\theta_2-\theta_1)=2\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_2-\theta_1}{2}.$$

Cancel the common factor $2\sin\frac{\theta_2-\theta_1}{2}$ from each denominator.

16. See Art. 45 (*a*), Note 2.

18. Draw roughly the lines, and find the co-ordinates of the vertices of the rectangle. Now find the equations of the diagonals (two-pt. form).

24. The equation of *any* line thro' the pt. of intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$ is

$$2x-3y+4+k(3x+4y-5)=0$$

or $\quad (2+3k)x+(4k-3)y+4-5k=0 \qquad \ldots(1)$

If it is ⊥ to the y-axis, its equation is of the form $x=a$,

∴ coeff. of $y=0$, i.e. $4k-3=0$, find the value of k. Substitute this value of k in (1).

25. Find the pt. of intersection of the last two lines.

Now find the equation of the line thro' the pt. of intersection of the first two lines and passing thro' the pt. of intersection of the last two lines, found above.

31. Inradius = length of the ⊥ from the incentre on any side, say, the first.

32. Use the *complete* ⊥ distance formula.

Note. *When the length of the ⊥ is given, use the complete ⊥ distance formula.* [Note 3 after Cor., Art. 57 (*b*)]

34. Let the required pt. be (x, y). Then $3x-2y-2=0 \qquad \ldots(1)$

$$\pm\frac{3x+4y-8}{5}=3 \qquad \ldots(2)$$

Solve (1) with each equation of (2).

35. The equations of the lines are $\quad x-y-4=0 \qquad \ldots(1)$

and $\quad 7x+y+20=0 \qquad \ldots(2)$

It will be found that the equation of the line thro' the origin and thro' the pt. of intersection of the lines (1) and (2) is

$$3x-y=0 \qquad \ldots(3)$$

To prove that the line (3) bisects the angle between the lines (1) and (2).

If θ is the *acute* angle between the lines (1) and (3), and ϕ the *acute* angle between the lines (2) and (3), show that $\tan\theta=\tan\phi$.

36. Let (x_1, y_1) be a pt. the ⊥s from which on the two lines are equal. Use the *complete* ⊥ distance formula. It will be found that

$$7x_1+4y_1-16=0,$$

∴ (x_1, y_1) lies on the line $7x+4y-16=0$, *i.e.* on the line

$$7x+4y=16.$$

37. Find the equations of the sides of the triangle, and show that each vertex and the origin are on the same side of the opposite side of the triangle.

CHAPTER IV

Page 94. Art. 62.

3. The equation is $x^2+3xy+4y^2-4x-6y+5=0 \qquad \ldots(1)$

Let (h, k) be the required co-ordinates of the pt.

Then, transferring the origin to (h, k), it will be found that (1) becomes

$$x^2+3xy+4y^2+x(2h+3k-4)+y(3h+8k-6)$$
$$+(h^2+3hk+4k^2-4h-6k+5)=0 \qquad \ldots(2)$$

If it does not contain terms of the first degree in x and y, then *coeff. of $x=0$, coeff. of $y=0$*,

i.e. $\quad 2h+3k-4=0,$

$\quad 3h+8k-6=0.$

Find the values of h and k.

Now (2) becomes $x^2+3xy+4y^2+(h^2+3hk+4k^2-4h-6k+5)=0$.

Substitute the values of h and k found above.

CHAPTER V

Page 96.

1. The equation is $x^2+2xy\sec\theta+y^2=0$. Solve it as a quadratic in x.

2. It will be found [by factorising the L.H.S.'s of the given equations] that the separate equations of the sides are

$$x=1, x=6\,;\ y=4, y=10.$$

Draw roughly the lines, and find the co-ordinates of the vertices of the rectangle. Now find the equations of the diagonals (two-pt. form).

3. It will be found that the separate equations of the lines are

$y=\frac{x}{\sqrt{3}}$, $y=\sqrt{3}\,x$, or $x-\sqrt{3}\,y=0$, $\sqrt{3}\,x-y=0$,

$\therefore$ the joint equation of the lines is $(x-\sqrt{3}\,y)(\sqrt{3}\,x-y)=0$.
[Rule (Art. 63, Note)]

Simplify.

Page 100.

1. Let the equation of a line thro' the origin be

$y=mx$ ...(1) | $mx-y=0$.

Then the distance of (x_1, y_1) from (1)$=\delta$ [Given]

$\therefore\ \pm\frac{mx_1-y_1}{\sqrt{m^2+1}}=\delta$ ...(2) [Complete $\perp$ distance formula]
[Note 3 after Cor., Art. 57 (*b*)]

Eliminate m from (1) and (2) [by substituting its value $\left(m=\frac{y}{x}\right)$ from (1) in (2)], and square.

2. Let the equation of *either* line of the pair

$ax^2+2hxy+by^2=0$ ...(1)

be $y=mx$...(2)

Then this value of y satisfies (1)

$\therefore\ a+2hm+bm^2=0$ ...(3) [As in solved Ex. 1, Art. 65]

It will be found that the equation of the line thro' (p, q) $\parallel$ to the line (2) is

$y-q=m(x-p)$ ...(4)

Eliminate m from (3) and (4) [by substituting its value $\left(m=\frac{y-q}{x-p}\right)$ from (4) in (3)].

3. Proceed as in solved Ex. 2, Art. 65.

4. The equation of the first pair of lines is

$ax^2+2hxy+by^2=0$ ...(1)

and that of the second pair is

$a'x^2+2h'xy+b'y^2=0$ ...(2)

Let the equation of one line of the pair (1) be

$y=mx$ ...(3)

Then this value of y satisfies (1). It will be found (as in solved Ex. 1, Art. 65) that

$bm^2+2hm+a=0$ ...(4)

The equation of the line thro' the origin $\perp$ to the line (3) is

$$y=-\frac{1}{m}x.$$

This is a line of the pair (2)

$\therefore$ this value of y satisfies (2). It will be found that

$a'm^2-2h'm+b'=0$ ...(5)

Eliminate m from (4) and (5). [Proceed as in solved Ex. 1, Art. 65.

5. The equation of the pair of lines is $ax^2+2hxy+by^2=0$.

Let the separate equations of the lines be

$y=m_1x$...(1), and $y=m_2x$...(2)

Then $m_1+m_2=-\frac{2h}{b}$, and $m_1m_2=\frac{a}{b}$ ...(3)
[Art. 65, Cor. 1]

The product of the $\perp$s from (x', y') on the lines (1) and (2)

$$=\frac{(m_1x'-y')}{\sqrt{m_1^2+1}}\cdot\frac{(m_2x'-y')}{\sqrt{m_2^2+1}}=\frac{m_1m_2x'^2-(m_1+m_2)x'y'+y'^2}{\sqrt{m_1^2m_2^2+m_1^2+m_2^2+1}}$$

$$=\frac{m_1m_2x'^2-(m_1+m_2)x'y'+y'^2}{\sqrt{m_1^2m_2^2+(m_1+m_2)^2-2m_1m_2+1}}\quad ...(4)$$

Substitute the values of m_1+m_2 and m_1m_2 from (3) in (4).

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1. (*a*) Proceed as in solved Ex., Art. 66.

2. (*i*) The equation is $x^2-3y^2=0$ ...(1)

Factorise the L.H.S. It will be found that (1) represents the two st. lines $x+\sqrt{3}\ y=0$, and $x-\sqrt{3}\ y=0$.

3. (*ii*) *Solve the equation as a quadratic in* x *thus* :

$$x=\frac{-24y\pm\sqrt{576y^2-176y^2}}{8}=\frac{-24y\pm20y}{8}=-\frac{y}{2},\ \text{or}\ -\frac{11}{2}y.$$

(*iii*) The equation is $x^2+2xy\cot 2\alpha-y^2=0$.

Solving it as a quadratic in x, it will be found that

$x+y(\cot 2\alpha-\operatorname{cosec} 2\alpha)=0$ ...(1)

or $x+y(\cot 2\alpha+\operatorname{cosec} 2\alpha)=0$ ...(2)

Now $\cot 2\alpha-\operatorname{cosec} 2\alpha$

$$=\frac{\cos 2\alpha}{\sin 2\alpha}-\frac{1}{\sin 2\alpha}=\frac{\cos 2\alpha-1}{\sin 2\alpha}=\frac{-(1-\cos 2\alpha)}{\sin 2\alpha}$$

$$=\frac{-2\sin^2\alpha}{2\sin\alpha\cos\alpha}=\frac{-\sin\alpha}{\cos\alpha}=-\tan\alpha.$$

Similarly it will be found that $\cot 2\alpha+\operatorname{cosec} 2\alpha=\cot\alpha$.

Substitute in (1) and (2).

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1. (*ii*) It will be found that the equation of the bisectors is

$\frac{x^2-y^2}{0}=\frac{xy}{-\sec\theta}$. Cross multiply. [Compare the Note in Hint to Ex. 4 (iii), Art. 41.]

2. The equation of the first pair of lines is

$$x^2-2pxy-y^2=0 \quad ...(1)$$

and that of the second pair is

$$x^2-2qxy-y^2=0 \quad ...(2)$$

It will be found that the equation of the bisectors of the angles between the pair (1) is

$$px^2+2xy-py^2=0 \quad ...(3)$$

If it is the same as the equation of the pair (2), then comparing coeffs. in (3) and (2),

$$\frac{p}{1}=\frac{2}{-2q}=\frac{-p}{-1}$$

or $pq=-1$.

The *symmetry* of this result in p and q shows that it is also the condition that the pair (1) should bisect the angles between the pair (2).

3. Find the equation of the bisectors of the angles between the first pair of lines, and show that it is the same as the equation of the bisectors of the angles between the second pair. Proceed as in solved Ex. 2, Art. 67.

Pages 110–111.

2. (*ii*) (*a*) Factorise. It will be found that the equation represents the two lines $x-3=0$, $y-2=0$.

(*c*) The first line is ∥ to the y-axis, and the second is ∥ to the x-axis. (*Draw roughly the lines.*)

∴ the angle between them $=90°$.

4. (*ii*) It will be found that $k=0$, or 1. If $k=0$, the given equation becomes $-4x+3y-12=0$ which, being of the first degree in x and y, represents *one* st. line (and *not* a pair of st. lines).

∴ *$k=0$ is rejected.*

7. It will be found that $\lambda=7$.

The lines are real or imaginary according as the lines thro' the origin ∥ to them are real or imaginary. Now the equation of the lines thro' the origin ∥ to the given lines is

$3x^2+9xy+7y^2=0$. [Second degree terms $=0$ (Art. 69, Cor. 1)]

Use Art. 65, Cor. 2.

8. The equation of the lines is

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let the separate equations of the lines be

$$lx+my+n=0 \quad ...(1), \text{ and } l'x+m'y+n'=0 \quad ...(2)$$

Then $ax^2+2hxy+by^2+2gx+2fy+c$

$=(lx+my+n)(l'x+m'y+n')$ identically

$=ll'x^2+(lm'+l'm)xy+mm'y^2$

$\quad+(ln'+l'n)x+(mn'+m'n)y+nn'$.

Equating coeffs. of x^2 on both sides, $ll'=a$ ...(3)

" " " xy " " " , $lm'+l'm=2h$ ...(4)

" " " y^2 " " " , $mm'=b$ ...(5)

" " " x " " " , $ln'+l'n=2g$ ...(6)

" " " y " " " , $mn'+m'n=2f$ ...(7)

" constant terms " " " , $nn'=c$ ...(8)

It will be found that the pt. of intersection of the lines (1) and (2) is $\left(\frac{mn'-m'n}{lm'-l'm}, \frac{nl'-n'l}{lm'-l'm}\right)$.

∴ square of the distance of this pt. from the origin (0, 0)

$$=\frac{(mn'-m'n)^2+(nl'-n'l)^2}{(lm'-l'm)^2}$$

$$=\frac{(mn'+m'n)^2-4mm'nn'+(nl'+n'l)^2-4nn'll'}{(lm'+l'm)^2-4ll'mm'}.$$

Substitute from (3), (4), (5), (6), (7), (8).

9. Proceed as in Ex. 8, and prove that square of the distance of the pt. of intersection from the origin

$$=\frac{c(a+b)-f^2-g^2}{ab-h^2} \quad ...(A)$$

If the lines are ⊥, $a+b=0$ or $a=-b$. Substitute in (A).

10. The equation is $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Let the separate equations of the two ∥ st. lines be

$$lx+my+n=0 \quad ...(1), \text{ and } lx+my+n'=0 \quad ...(2)$$

Then $ax^2+2hxy+by^2+2gx+2fy+c$

$=(lx+my+n)(lx+my+n')$ identically

$=l^2x^2+2lmxy+m^2y^2+l(n'+n)x+m(n'+n)y+nn'$.

Equate coeffs. of like terms on both sides. [Proceed as in Ex. 8.]

It will be found that $l^2=a$ ...(3), $2lm=2h$ ...(4), $m^2=b$ ...(5)

$l(n'+n)=2g$ ...(6), $m(n'+n)=2f$ ...(7), $nn'=c$ ...(8)

From (4), $h^2=l^2m^2$, also from (3) and (5), $ab=l^2m^2$

∴ $h^2=ab$.

Again from (5) and (6), $bg^2=\frac{m^2l^2(n'+n)^2}{4}$,

also from (3) and (7), $af^2=\frac{l^2m^2(n'+n)^2}{4}$

∴ $bg^2=af^2$.

It will be found that the distance between the ∥ lines (1) and (2)

$$=\frac{n'-n}{\sqrt{l^2+m^2}}=\frac{\sqrt{(n'+n)^2-4nn'}}{\sqrt{l^2+m^2}} \quad ...(9)$$

From (8), $(n'+n)^2=\frac{4g^2}{l^2}$ [But from (1), $l^2=a$]

$$=\frac{4g^2}{a} \quad ...(10)$$

Substitute from (10), (8), (3), (5) in (9).

Pages 112—113, Art. 70.

4. The equations of the curves are

$$ax^2+2hxy+by^2+2gx=0 \quad ...(1)$$

and $$a'x^2+2h'xy+b'y^2+2g'x=0 \quad ...(2)$$

To make (1) homogeneous by means of (2), eliminate the term in x from (1) and (2) [by multiplying (1) by g', (2) by g, and substracting].

6. The equation of the line is $x-y-1=0$ ...(1)
and that of the curve is $5x^2+12xy-6y^2+4x-2y+3=0$ ...(2)

It will be found that the equation of the lines joining the origin to the pts. of intersection of the line (1) and the curve (2) is

$12x^2-y^2=0$ or $y^2=12x^2$ or $y=\pm 2\sqrt{3}\, x$,

i.e. $y=2\sqrt{3}\, x$,

or $y=-2\sqrt{3}\, x$.

Here $m_1=2\sqrt{3}$, $m_2=-2\sqrt{3}$, $\therefore m_1=-m_2$.

$\therefore$ these lines make equal angles with the x-axis.

$\therefore$ they make equal angles with the y-axis also.

MISCELLANEOUS EXAMPLES ON CHAPTER V

Pages 113—115.

3. [To find the equations of the two lines thro' the origin making an angle α with $x+y=0$.]

The equation of *any* line thro' the origin is $y=mx$ ...(1)

If it makes an angle α with $x+y=0$ (slope$=-1$),

then $$\tan\alpha=\pm\frac{m-(-1)}{1+m(-1)}=\pm\frac{m+1}{1-m} \quad ...(2)$$

Eliminating m from (1) and (2) [by substituting its value $\left(m=\frac{y}{x}\right)$ from (1) in (2)],

$$\tan\alpha=\pm\frac{y+x}{x-y} \text{ or } \frac{\sin\alpha}{\cos\alpha}=\pm\frac{x+y}{x-y} \text{ or, squaring, } \frac{\sin^2\alpha}{\cos^2\alpha}=\frac{(x+y)^2}{(x-y)^2}.$$

4. [To find the equations of the two lines thro' the origin making an angle of 60° with the line $ax+by+c=0$.]

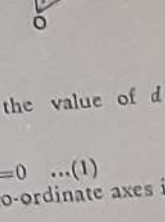
The equation of *any* line thro' the origin is $y=mx$ ...(1)

If it makes an angle of 60° with $ax+by+c=0$ $\left(\text{slope}=-\frac{a}{b}\right)$,

then $$\tan 60^\circ=\pm\frac{m-\left(-\frac{a}{b}\right)}{1+m\left(-\frac{a}{b}\right)}=\pm\frac{bm+a}{b-am} \quad ...(2)$$

Eliminating m from (1) and (2) [by substituting its value $\left(m=\frac{y}{x}\right)$ from (1) in (2)],

$\sqrt{3}=\pm\frac{by+ax}{bx-ay}$ or, squaring, $(ax+by)^2-3(bx-ay)^2=0$.



These lines OA, OB make $\angle$s of 60° with the line $ax+by+c=0$, *i.e.* with AB.

$\therefore$ they form an equilateral $\triangle$ with it.

The length of the $\perp$ OC from O (0, 0) on AB is

$$d=\frac{c}{\sqrt{a^2+b^2}}.$$

Now OC bisects AB

$\therefore$ AB$=2$AC$=2d\cot 60^\circ=\frac{2d}{\sqrt{3}}$.

$\therefore \triangle$OAB$=\frac{1}{2}$AB. $d=\frac{1}{2}.\frac{2d}{\sqrt{3}}.d=\frac{d^2}{\sqrt{3}}$. Substitute the value of d found above.

5. The equation of the lines is $ax^2+2hxy+by^2=0$ ...(1)

One of them will bisect an angle between the co-ordinate axes if its equation is $y=x$, or $y=-x$,

i.e. if these values of y satisfy (1),

or if $a+2h+b=0$, $a-2h+b=0$

or if $a+b=-2h$ ...(2), $a+b=2h$ ...(3)

or if, combining (2) and (3), $a+b=\pm 2h$

or if, squaring, $(a+b)^2=4h^2$.

6. The equation is $\lambda y^2+(1-\lambda^2)xy-\lambda x^2=0$,

or $\lambda y^2+xy-\lambda^2xy-\lambda x^2=0$.

Factorise.

7. The equation is

$$2(x+2)^2+3(x+2)(y-2)-2(y-2)^2=0 \quad ...(1)$$

Transferring the origin to $(-2, 2)$,* by putting

$x=x'-2$, $y=y'+2$ ...(2) [Art. 62]

*How to write this step. In (1), putting $x+2=0$, $y-2=0$, we get $x=-2$, $y=$
$\therefore$ we transfer the origin to $(-2, 2)$.

(1) becomes $2x'^2+3x'y'-2y'^2=0$

[Factorise]

or $2x'^2+4x'y'-x'y'-2y'^2=0$ or $2x'(x'+2y')-y'(x'+2y')=0$

or $(x'+2y')(2x'-y')=0$ or $x'+2y'=0$ ...(3), $2x'-y'=0$ ...(4)

$\therefore$ the equation represents the two st. lines $y'=-\frac{1}{2}x'$, $y'=2x'$.

Here $m_1=-\frac{1}{2}$, $m_2=2$, $\therefore m_1m_2=-\frac{1}{2}(2)=-1$

$\therefore$ the lines are $\perp$.

The pt. of intersection of the lines (3) and (4) is $x'=0$, $y'=0$

or from (2), $x=-2$, $y=2$, $\therefore$ the pt. of intersection is $(-2, 2)$.

8. The equation of the line-pair is

$$(A-h)x^2+2axy+(A+h)y^2=0 \quad ...(1)$$

Find the equation of the angle-bisectors.

9. Use the method of "solving as quadratic". Proceed as in solved Ex. 1, Art. 69.

13. Proceed as in solved Ex. 2, Art. 69.

16. The equation of the lines thro' the origin $\parallel$ to the lines

$$2x^2+5xy+2y^2+10x+5y=0$$

is $\quad 2x^2+5xy+2y^2=0.\quad$ [Second degree terms $=0$]

[Art. 69, Cor. 1]

Now find the equation of the lines thro' the origin $\perp$ to these.

[Proceed as in solved Ex. 2, Art. 65.]

17. It will be found that the separate equations of the angle-bisectors of the lines are

$$\frac{x-4y+2}{\sqrt{17}}=\pm\frac{x-y-1}{\sqrt{2}}.$$

Squaring both sides, the joint equation of the angle-bisectors is

$$\frac{(x-4y+2)^2}{17}=\frac{(x-y-1)^2}{2}$$

or $\quad 2(x-4y+2)^2=17(x-y-1)^2.$

Use $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$, and simplify.

18. Proceed as in Ex. 8, Art. 69.

23. The equation of the line is $lx+my=1$ ...(1)

and that of the curve is $\quad x^2+y^2=a^2$ ...(2)

It will be found that the equation of the lines joining the origin to the pts. of intersection of the line (1) and the curve (2) is

$$x^2(a^2l^2-1)+2a^2lmxy+y^2(a^2m^2-1)=0.$$

[Compare with $ax^2+2hxy+by^2=0$]

If the angle between these lines $=45°$,

then $\tan 45°=\dfrac{2\sqrt{h^2-ab}}{a+b}=\,?$

Square.

CHAPTER VI

Page 117.

3. (*i*) The centre of the circle is (2, 3), and radius = distance between (2, 3) and (5, 7) = ?

$\therefore$ the equation of the circle is ?

Page 120.

2. (*i*) The equation is $a(x^2+y^2)+2gx+2fy+c=0$.

Divide thro'out by a, and proceed as in Art. 74.

4. Find the centre, and plot it. Notice that the circle passes thro' the *origin* (put $x=0$, $y=0$ in the equation)

$\therefore$ *radius = join of the centre to the origin.*

Pages 121—122.

1. Choose the axes as in Art. 10 (*b*).

3. The equation of the line is

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(1)$$

where $\quad \dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c^2} \quad ...(2)$

It will be found that the equation of the line, $\perp$ to the line (1) and passing thro' the origin (0, 0), is

$$\frac{x}{b}-\frac{y}{a}=0 \quad ...(3)$$

To find the locus of the foot of the $\perp$, *i.e.* the locus of the pt. of intersection of the lines (1) and (3), we have to eliminate a, b from (1), (2), (3).

Squaring (1) and (3), and adding,

$$x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+y^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1 \quad ...(4)$$

Substitute from (2) in (4).

Page 123.

3. The equations of the lines are

$x+y-6=0$ ...(1)

$2x+y-4=0$ ...(2)

$x+2y-5=0$ ...(3)

Take the equations of the lines two at a time, solve them simultaneously as in Algebra, and find the co-ordinates of the vertices of the triangle.

Now proceed as in solved Ex., Art. 76.

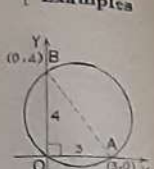
4. Proceed as in solved Ex., Art. 76.

Also $\because$ the centre $(-g, -f)$ lies on the line $4x+y=16$

$\therefore -4g-f=16.$

Page 124.

1. (*b*) Proceed as in Art. 77.



3. $\because \angle$ at $O = 1$ rt. $\angle$

$\therefore$ the circle has the join of $(3, 0)$ and $(0, 4)$ as diameter.

Page 128.

4. The equation of the circle is $x^2+y^2=10$. Putting $x=1$, $1+y^2=10$ or $y^2=9$, $\therefore y=\pm 3$, $\therefore$ the pts. are $(1, 3)$ and $(1, -3)$.

5. (*a*) **[Method of chord.]**

The equation of the circle is $x^2+y^2+2gx+2fy+c=0$.

Let (x', y') be any other pt. on the circle. Proceed as in Art. 81.

(*b*) The equations of the circles are $x^2+y^2-4x+6y+8=0$ ...(1)

and $x^2+y^2-10x-6y+14=0$ ...(2)

Show that the pt. $(3, -1)$ lies on both the circles (1) and (2), and find the equations of the tangents to the circles at this pt.

Show that these equations reduce to the same.

Page 131.

3. The equation of the line is $y=mx+c$.

If it is a normal to the circle whose centre is (a, b) and radius r, it passes thro' the centre of the circle (Art. 83, Cor.), $\therefore$?

Pages 133—134.

2. Length of the chord

$=2\sqrt{(\text{radius})^2-(\text{central} \perp)^2}$,

where central $\perp$ means the $\perp$ from the centre on the line.

3. Proceed as in Ex. 2.

4. Length of the chord (intercept)

=distance between the pts. of intersection.

5. (*b*) The equation of the circle is $x^2+y^2=a^2$ ...(1)

nd that of the line is $y=px+q$ ...(2)

Substituting the value of y from (2) in (1), it will be found that

$x^2(1+p^2)+2pqx+(q^2-a^2)=0$ ... ...(3)

hich is a quadratic in x. Let x_1, x_2 be the roots.

Then the x-coordinate of the mid-pt. of the chord is

$$x=\frac{x_1+x_2}{2}$$

[But x_1+x_2=sum of the roots of quadratic (3)

$$=-\frac{2pq}{1+p^2}\Big]$$

$$=\tfrac{1}{2}\left[-\frac{2pq}{1+p^2}\right]=-\frac{pq}{1+p^2}.$$

$\because$ the mid-pt. of the chord lies on the line (2),

$\therefore$ substituting the above value of $x\left(=-\frac{pq}{1+p^2}\right)$ in (2),

$$y=p\left(-\frac{pq}{1+p^2}\right)+q=\frac{q}{1+p^2}.$$

$\therefore$ the mid-pt. of the chord is $\left[-\frac{pq}{1+p^2}, \frac{q}{1+p^2}\right]$.

Note. The above method of finding the mid-pt. of the chord intercepted by a circle on a line is applicable to other curves in later chapters as well.

6. Proceed as in Ex. 5.

Pages 135—136.

1. (*a*) Proceed as in Art. 86, and get the equation (3), which is a quadratic in x giving two values of x.

Substituting these values of x one by one in (1), we get the corresponding values of y.

These corresponding values of x and y give us two pts.

$\therefore$ the line meets the circle in two pts.

(*b*) If the line *touches* the circle, the quadratic (3) has equal roots.

4. Proceed as in solved Ex. 2, Art. 88.

5. Proceed as in solved Ex. 2, Art. 88.

Page 137.

1. (*i*) Use the *tangent property*: **If a line touches a circle, the length of the $\perp$ from the centre on the line=the radius.**

3. Radius of the circle

=length of the $\perp$ from the centre on the line.

5. (*ii*) The equation of the circle is $x^2+y^2=4$ ...(1)

The equation of *any* line $\parallel$ to the y-axis is $x=a$ ...(2)

If it touches the circle (1),

the length of the $\perp$ from the centre $(0, 0)$ on it=the radius

$\therefore$? Find the values of a.

Substitute these values of a one by one in (2).

6. Use the method of solved Ex. 2, Art. 89.

Page 141.

1. To find the co-ordinates of the pts. of contact, proceed as in steps (*i*) and (*ii*) of solved Ex. 2, Art. 88.

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2. The equation of the circle is $x^2+y^2+2gx+2fy+c=0$ ...(1)

Proceed as in Art. 92.

4. It will be found that the equation of the pair of tangents from the origin (0, 0) is

$$(x^2+y^2+2gx+2fy+c)c = (gx+fy+c)^2 \qquad [\, SS_1=T^2 \,]$$

or $$c(x^2+y^2)+2gcx+2fcy+c^2 = (gx+fy)^2+2c(gx+fy)+c^2$$

[Cancel common terms on both sides]

or $$c(x^2+y^2)=(gx+fy)^2.$$

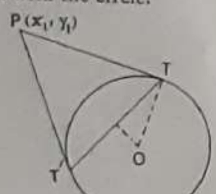
Page 144.

1. Proceed as in Art. 96.

2. The equation of the circle is $x^2+y^2=25$ ...(1)

Find the equation of the chord of contact of tangents from (7, 1) to the circle (1) and then its pts. of intersection with the circle.

3. The equation of the circle is $x^2+y^2=a^2$.



Let P be the pt. (x_1, y_1), and PT, PT′ the tangents from P to the circle, T, T′ being the pts. of contact.

Then the equation of TT′, the chord of contact of tangents from P to the circle, is

$$xx_1+yy_1=a^2$$

or $$xx_1+yy_1-a^2=0.$$

∴ length of the chord TT′

$$=2\sqrt{a^2-p^2} \quad \left[\text{But } p=\frac{a^2}{\sqrt{x_1^2+y_1^2}}\right]$$

$$=2\sqrt{a^2-\frac{a^4}{x_1^2+y_1^2}}$$

$$=\frac{2a\sqrt{x_1^2+y_1^2-a^2}}{\sqrt{x_1^2+y_1^2}}$$

∴ $\Delta PTT'=\frac{1}{2}TT'$. ($\perp$ from P on TT′)

$$=\frac{1}{2}\frac{2a\sqrt{x_1^2+y_1^2-a^2}}{\sqrt{x_1^2+y_1^2}}\cdot\frac{x_1^2+y_1^2-a^2}{\sqrt{x_1^2+y_1^2}}=a\frac{(x_1^2+y_1^2-a^2)^{\frac{3}{2}}}{x_1^2+y_1^2}.$$

4. Proceed as in Art. 96, using the equation of the tangent und in Art. 81.

Page 145.

1. Proceed as in Art. 98, using the equation of the chord of ntact found in Ex. 4, Art. 96.

Page 149.

3. Let (x_1, y_1) be the pole of the line w.r.t. the circle

$$x^2+y^2=a^2 \quad ...(1)$$

∵ (x_1, y_1) lies on the circle $x^2+y^2=9a^2$, ∴ $x_1^2+y_1^2=9a^2$ ...(2)

The equation of the line [*i.e.* polar of (x_1, y_1) w.r.t. the circle (1)] is

$$xx_1+yy_1=a^2.$$

It touches the circle $9x^2+9y^2=a^2$ or $x^2+y^2=\frac{a^2}{9}$,

if (use the tangent property)

$$\frac{a^2}{\sqrt{x_1^2+y_1^2}}=\frac{a}{3} \quad \text{or} \quad x_1^2+y_1^2=9a^2,$$

which is true from (2).

Page 151.

1. Eliminate θ by squaring the two equations and adding.
2. *Any* pt. on the circle $x^2+y^2=a^2$ is $(a\cos\theta, a\sin\theta)$.
3. The equation of the circle is $x^2+y^2=a^2$.

Let $P(a\cos\alpha, a\sin\alpha)$, $Q(a\cos\beta, a\sin\beta)$ be the extremities of the chord, and C the pt. $(c, 0)$.

Then the slope of CP $=\dfrac{a\sin\alpha}{a\cos\alpha-c}$.

Similarly the slope of CQ $=\dfrac{a\sin\beta}{a\cos\beta-c}$.

But CP is $\perp$ to CQ,

$$\therefore \frac{a\sin\alpha}{a\cos\alpha-c}\cdot\frac{a\sin\beta}{a\cos\beta-c}=-1$$

or $$a^2\sin\alpha\sin\beta=-[a^2\cos\alpha\cos\beta-ac(\cos\alpha+\cos\beta)+c^2]$$

or $$a^2\cos(\alpha-\beta)-ac(\cos\alpha+\cos\beta)+c^2=0... \qquad ...(1)$$

Now the co-ordinates of the mid-pt. of PQ are

$$x=\frac{a\cos\alpha+a\cos\beta}{2}=a\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad ...(2)$$

$$y=\frac{a\sin\alpha+a\sin\beta}{2}=a\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad ...(3)$$

To find the locus of the mid-pt. of PQ we have to eliminate α, β from (1), (2), (3).

Squaring (2) and (3), and adding,

$$x^2+y^2=a^2\cos^2\frac{\alpha-\beta}{2}\left[\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right]=a^2\cos^2\frac{\alpha-\beta}{2} \qquad ...(4)$$

From (1), $$a^2\left[2\cos^2\frac{\alpha-\beta}{2}-1\right]-ac.2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+c^2=0 \qquad ...(5)$$

Substitute from (4) and (2) in (5).

Pages 154—155.

2. Let the equation of the circle be $x^2+y^2=a^2$ or $x^2+y^2-a^2=0$.

[Form : S=0]

Let (x_1, y_1) be the mid-pt. of the chord.

Then the equation of the chord is $xx_1+yy_1-a^2=x_1^2+y_1^2-a^2$

or $$xx_1+yy_1=x_1^2+y_1^2. \qquad [\,T=S_1\,]$$

Find the slope of the chord, also the slope of the line passing thro' the centre (0, 0) of the circle and the mid-pt. (x_1, y_1) of the chord (Art. 41, Cor.).

3. The equation of the circle is $x^2+y^2=a^2$.
Let (x_1, y_1) be the mid-pt. of *any* chord.
Then it will be found that the equation of the chord is
$$xx_1+yy_1=x_1^2+y_1^2.$$
If it passes thro' (h, k), then $hx_1+ky_1=x_1^2+y_1^2$
$\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y)]
$$hx+ky=x^2+y^2.$$

4. Proceed as in Art. 103.

Pages 156—157.

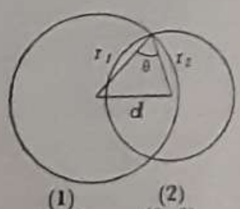
1. (*a*) Proceed as in Art. 105.
4. Let (x_1, y_1) be *any* pt.
5. Let (x_1, y_1) be the pt. the tangents from which to the three rcles are equal in length. Then
$$\sqrt{x_1^2+y_1^2-4x_1+7}=\sqrt{x_1^2+y_1^2-\text{[illegible]}x_1+\text{[illegible]}y_1+\text{[illegible]}}=\sqrt{x_1^2+y_1^2+y_1}.$$
Square, and cancel $x_1^2+y_1^2$ from each member. Solve for x_1, y_1.

Page 158.

3. Let the required equation of the circle be
$$x^2+y^2+2gx+2fy+c=0\ldots \qquad \ldots(1)$$
$\because$ it passes thro' (0, 0), $\therefore c=0 \quad \ldots(2)$
$\because$ it cuts $x^2+y^2-8y+12=0$ orthogonally,
$\therefore 2g(0)+2f(-4) \qquad =c+12 \quad \ldots(3) \quad [\,2g_1g_2+2f_1f_2=c_1+c_2\,]$
Again $\because$ it cuts $x^2+y^2-4x-6y-3=0$ orthogonally,
$\therefore 2g(-2)+2f(-3)=c-3 \quad \ldots(4)$
Solve (2), (3), (4) for g, f, c, and substitute these values in (1).

Page 159.

. It will be found that the centres of rcles are (1, 2) and (0, 0), eir radii are 3 and 4.



(1) (1, 2) (2) (0, 0)

$r_1=3,\ r_2=4$;
the distance between the centres,
$$=\sqrt{(0-1)^2+(0-2)^2}=\sqrt{5}.$$

Now $\cos\theta=\dfrac{r_1^2+r_2^2-d^2}{2r_1r_2}=?$

Page 164.

2. (*b*) Find the length of the intercept made by the first circle on the radical axis (common chord) by the method of Note, Art. 87. (Use the intercept formula.)

5. (*i*) Find the equation of the radical axis of the two circles.
(*ii*) Find the condition that this radical axis may touch one of the two circles, say, the first.

7. The equation of the common tangent of two touching circles at the pt. of contact is the equation of their radical axis.

8. (*a*) Show that distance between the centres
=difference of the radii.
(*b*) The pt. of contact divides the join of the centres *externally* in the ratio of the radii.

Pages 166—167.

2. The equations of the circles are
$$x^2+y^2-1=0 \quad \ldots(1)$$
$$x^2+y^2-4x+6y+2=0 \quad \ldots(2)$$
$$x^2+y^2-8x+12y+1=0 \quad \ldots(3)$$
It will be found that the equation of the radical axis of the circles (1) and (2) is
$$4x-6y-3=0\ldots \qquad \ldots(4)$$
and the radical axis of the circles (2) and (3) is
$$4x-6y+1=0\ldots \qquad \ldots(5)$$
Solving (4) and (5) by cross-multiplication,
$$\frac{x}{-24}=\frac{y}{-16}=\frac{1}{0}, \quad \therefore x=\frac{-24}{0},\ y=\frac{-16}{0}$$
$\therefore x, y$ are both infinite.

Page 169.

1. Proceed as in Art. 114.

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1. The equations of the circles are
$$x^2+y^2+2x+3y-7=0 \quad \ldots(1)$$
$$x^2+y^2+3x-2y-1=0 \quad \ldots(2)$$
The equation of their common chord (radical axis) is
[Subtracting (2) from (1)]
$$-x+5y-6=0$$
or $$x-5y+6=0\ldots \qquad \ldots(3)$$

The equation of *any* circle thro' the pts. of intersection of the circles (1) and (2), *i.e.* thro' the pts. of intersection of the circle (1) and the common chord (3) is

$$x^2+y^2+2x+3y-7+k(x-5y+6)=0 \quad ...(4)$$

If it passes thro' (1, 2), then ? Find the value of k.

Substitute this value of k in (4).

4. The equations of the circles are

$$x^2+y^2-4x-6y-12=0 \quad ...(1)$$
$$x^2+y^2+6x+4y-12=0 \quad ...(2)$$

The equation of their common chord (radical axis) is [Subtracting (2) from (1)]

$$-10x-10y=0 \text{ or } x+y=0 \quad ...(3)$$

(*i*) The equation of *any* circle thro' the pts. of intersection of the circles (1) and (2), *i.e.* thro' the pts. of intersection of the circle (1) and the common chord (3) is

$$x^2+y^2-4x-6y-12+k(x+y)=0 \quad ...(4)$$

(*ii*) If its centre $[-\frac{1}{2}(-4+k), -\frac{1}{2}(-6+k)]$ lies on the common chord (3), find the value of k.

(*iii*) Substitute this value of k in (4).

5. The equations of the circles are $x^2+y^2-1=0 \quad ...(1)$

$$x^2+y^2-2x-4y+1=0 \quad ...(2)$$

Proceed as in Ex. 4. It will be found that the equation of their common chord (radical axis) is $x+2y-1=0 \quad ...(3)$

The equation of *any* circle thro' the pts. of intersection of the circles (1) and (2), *i.e.* thro' the pts. of intersection of the circle (1) and the common chord (3) is

$$x^2+y^2-1+k(x+2y-1)=0 \quad ...(4)$$

If it touches the line $x+2y=0$, use the tangent property, and find the value of k.

Substitute this value of k in (4).

Page 171.

1. Let the equations of the two circles be

$$x^2+y^2+2g_1x+c=0 \quad ...(1)$$
$$x^2+y^2+2g_2x+c=0 \quad ...(2) \qquad \text{[Art. 116, Note]}$$

Let the equation of the third circle be

$$x^2+y^2+2gx+2fy+d^*=0 \quad ...(3)$$

If the circle (1) cuts the circle (3) orthogonally, then

$$2g_1g+2(0)f=c+d \quad ...(4) \qquad [\, 2g_1g_2+2f_1f_2=c_1+c_2 \,]$$

*Why to use d (and not c). In order to avoid confusion with c in the equation of circle (1) or (2), we take the equation of the third circle as $x^2+y^2+2gx+2fy+d=0$ (and *not* as $x^2+y^2+2gx+2fy+c=0$).

Again, if the circle (2) cuts the circle (3) orthogonally, then

$$2g_2g+2(0)f=c+d \quad ...(5)$$

Subtracting (5) from (4), $2g(g_1-g_2)=0$ or $g=0 \quad ...(6)$

The equation of the radical axis of the circles (1) and (2) is

$$x=0 \quad ...(7)$$

Substituting the co-ordinates of the centre $(-g, -f)$ of the circle (3) in (7), we get

$$-g=0 \text{ or } g=0, \text{ which is true from (6)}$$

$\therefore$ the radical axis of the circles (1) and (2) passes thro' the centre of the circle (3).

2. Proceed as in Ex. 1, and get $g=0 \quad ...(6)$

Now the co-ordinates of the centre of the circle (3) are

$$x=-g, y=-f \quad ...(7)$$

Eliminating g, f from (6) and (7),

the locus of the centre is $-x=0$ or $x=0$,

which is the radical axis of the circles (1) and (2).

Page 172.

1. The equation of *any* circle whose radical axis in relation to $x^2+y^2-a^2=0$ is $x-\frac{a}{2}=0$, is

$$x^2+y^2-a^2+k\left(x-\frac{a}{2}\right)=0 \quad ...(1)$$

If it passes thro' $(2a, 0)$, then ? Find the value of k.

Substitute this value of k in (1).

2. (*a*) Choose the axes as in Art. 10 (*b*).

(*i*) Let $P(x, y)$ be *any* pt. on the locus.

(*ii*) Then $PA=n.PB$.

(*iii*) It will be found that the locus of P is

$$x^2+y^2+2a\frac{1+n^2}{1-n^2}x+a^2=0 \quad ...(1)$$

which is a circle.

(*b*) For different values of n, let the equations of *any* pair of the system be [putting $n=n_1, n_2$ in (1)],

$$x^2+y^2+2a\frac{1+n_1^2}{1-n_1^2}x+a^2=0 \quad ...(2)$$
$$x^2+y^2+2a\frac{1+n_2^2}{1-n_2^2}x+a^2=0 \quad ...(3)$$

The equation of their radical axis is [Subtracting (3) from (2)]

$$2a\left[\frac{1+n_1^2}{1-n_1^2}-\frac{1+n_2^2}{1-n_2^2}\right]x=0 \text{ or } x=0,$$

which is independent of n_1, n_2.

$\therefore$ *any* pair of the system of circles (1) for different values of n has the same radical axis.

Pages 171—174.

1. Let the equations of the given circles be

$$x^2+y^2+2g_1x+c=0 \quad ...(1)$$
$$x^2+y^2+2g_2x+c=0 \quad ...(2) \quad \text{[Art. 116, Note]}$$

2. The equations of the circles are

$$x^2+y^2-4=0 \quad ...(1)$$
$$x^2+y^2+2x+4y-6=0 \quad ...(2)$$

The equation of their radical axis is [Subtracting (2) from (1)]

$$-2x-4y+2=0 \text{ or } x+2y-1=0.$$

$\therefore$ the equation of *any* circle coaxal with the circles (1) and (2) is

$$x^2+y^2-4+k(x+2y-1)=0 \quad ...(3)$$

Page 176, Art. 119.

2. The equations of the circles are

$$x^2+y^2+2x=0 \quad ...(1)$$
$$x^2+y^2-6x=0 \quad ...(2)$$

Proceed as in solved Ex., Art. 119. It will be found that the equations of the direct common tangents are

$$x+\sqrt{3}\,y+3=0 \quad ...(3)$$
$$x-\sqrt{3}\,y+3=0 \quad ...(4)$$

[To find the equations of the tangents from (0, 0) to the circle (1).]

The equation of *any* line thro' (0, 0) is $y=mx$ ...(5)

If it touches the circle (1) (use the tangent property), it will be found that

$$\pm\frac{-m}{\sqrt{m^2+1}}=1 \text{ or, squaring, } m^2=m^2+1$$

or $0.m^2-1=0$ [*Note this step*]

$\because$ coeff. of $m^2=0$, one root is infinite.

Now from (5), $x=\dfrac{y}{m}$

$\therefore$ when m becomes infinite, $x=0$ ...(6)

which is the equation of the transverse common tangent.

[To prove that the tangents (3), (4), (6) form an equilateral $\triangle$.]

Use the angle formula $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$, and prove that the acute angle between the tangents (3) and (4) $=60^\circ$.

Use the angle formula (Second form)

$$\tan\theta=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2} \quad \text{[Art. 45 (b)]},$$

and prove that the acute angle between the tangents (4) and (6) $=60^\circ$.

3. "Find the common external tangents" means "find the *equations* of the direct common tangents."

MISCELLANEOUS EXAMPLES ON CHAPTER VI

Pages 176—179.

1. Draw a rough Fig. Join the centre of the circle to its pts. of contact with the axes. The centre is (2, 2), and the radius $=2$.

4. Let the fixed pts. be (x_1, y_1), (x_2, y_2), (x_3, y_3).

7. Eliminate α (see Rule in Note after solved Ex., Art. 75) [by squaring the two equations and adding].

16. Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad ...(1)$$

Its centre is $(-g, -f)$, and radius $=\sqrt{g^2+f^2-c}$.

$\because$ the circle (1) touches the x-axis $(y=0)$,

the length of the $\perp$ from the centre on the x-axis = the radius

$$\therefore \pm\frac{(-f)}{1}=\sqrt{g^2+f^2-c},$$

or, squaring, $\quad f^2=g^2+f^2-c$

or $\quad g^2-c=0$ ...(2)

Again $\because$ it touches the y-axis $(x=0)$,

similarly $\quad f^2-c=0$ ...(3)

Also $\because$ its radius $=a$

$$\therefore \sqrt{g^2+f^2-c}=a$$

or, squaring, $g^2+f^2-c=a^2$ ...(4)

Substitute the values of g^2 and f^2 from (2) and (3) in (4).

It will be found that $c=a^2$.

Substitute this value of $c(=a^2)$ in (2), and then in (3).

It will be found that $g=\pm a$, $f=\pm a$.

Substitute these values of g, f, c in (1).

17. (*i*) Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+d^*=0 \quad ...(1)$$

Proceed as in Ex. 16. It will be found that

$$f^2-d=0 \quad ...(2)$$
$$g^2-d=0 \quad ...(3)$$
$$2cg+c^2=f^2-d \quad ...(4)$$

---

*Why to use d (and not c). In order to avoid confusion with the given c in $x=c$, we take the equation of the circle as $x^2+y^2+2gx+2fy+d=0$ (and *not* as $x^2+y^2+2gx+2fy+c=0$).

Substituting the value of $f^2(=d)$ from (2) in (4),

$$g=-\frac{c}{2}.$$

Substituting this value of g in (3),

$$d=\frac{c^2}{4}.$$

Substituting this value of d in (2),

$$f=\pm\frac{c}{2}.$$

Substitute these values of g, f, d in (1).

(*ii*) Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad ...(1)$$

Proceed as in Ex. 16. It will be found that

$$f^2-c=0 \qquad ...(2)$$

$$2ga+a^2=f^2-c \qquad ...(3)$$

$$\left(\frac{-3g-4f+5a}{5}\right)^2=g^2+f^2-c \quad ...(4)$$

Substituting the value of $c(=f^2)$ from (2) in (3), $g=-\frac{a}{2}$.

Substitute the values of $c(=f^2)$ from (2), and $g=-\frac{a}{2}$ (found above) in (4).

It will be found that $\left(\frac{\frac{13a}{2}-4f}{5}\right)^2=\frac{a^2}{4}$

or, extracting square root, $\frac{\frac{13a}{2}-4f}{5}=\pm\frac{a}{2}$.

Find the values of f. Substitute these values of f one by one in (2), and find the corresponding values of c.

Substitute these corresponding values of g, f, c in (1).

18. (*i*) Show that the length of the $\perp$ from the centre of the circle on the line, *in magnitude*, = the radius.

19. The equation of the circle is $x^2+y^2=25$.

[Compare with $x^2+y^2=a^2$]

Here $a^2=25$, $\therefore a=5$.

Let m be the slope of a tangent.

Then the equation of the tangent is

$$y=mx\pm5\sqrt{1+m^2}. \qquad [\, y=mx\pm a\sqrt{1+m^2}\,]$$

Now $m=\tan 0°$, $\tan 60°$, $\tan 120°$.

24. The equation of the circle is $x^2+y^2=9$ ...(1)

PQ is the polar of the pt. (5, 4) w.r.t. the circle (1).

[Art. 98, Note (*i*)]

25. The equation of the circle is

$$x^2+y^2=a^2 \quad ...(1)$$

The equation of the chord of contact of tangents from (x', y') to the circle (1) is $xx'+yy'=a^2$ or $\frac{xx'+yy'}{a^2}=1$ ...(2)

Find the equation of the lines joining the origin to the pts. of intersection of the circle (1) and the chord of contact (2).

[Rule of Note, Art. 70]

Find the condition that these lines should be $\perp$. (Transpose, and use coeff. of x^2+coeff. of $y^2=0$.)

28. Let P be the pt. (x_1, y_1).

30. The equation of the circle is $x^2+y^2=a^2$ ...(1)

Find the pole of the line $x\cos\alpha+y\sin\alpha=p$ w.r.t. the circle (1).

31. Let the equation of the circle be $x^2+y^2=a^2$.

Let (x_1, y_1) be the mid-pt. of *any* chord. It will be found that the equation of the chord is

$$xx_1+yy_1=x_1^2+y_1^2.$$

Its length (intercept) $=2\sqrt{a^2-p^2}$, which will be found $=2\sqrt{a^2-(x_1^2+y_1^2)}$.

If this length $=2k$, then ? Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

33. (*a*), (*b*), (*c*) Proceed as in solved Ex. 2, Art. 111.

(*d*) Find the centres of the circles. It will be found that the centres are on opposite sides of the tangent at the pt. of contact.

$\therefore$ the contact is external.

34. The equations of the circles are

$$x^2+y^2-12=0 \quad ...(1)$$

$$x^2+y^2-5x+3y-2=0 \quad ...(2)$$

It will be found that the equation of their common chord (radical axis) is $5x-3y-10=0$ ...(3)

Find the pole of the common chord (3) w.r.t. the circle (1).

CHAPTER VII

Page 182.

1. The equation of the parabola is $y^2=8x$ ...(1)

In (1), put $y=2x$ ...(2), and find the values of x.

Substitute these values of x one by one in (2), and find the corresponding values of y.

2. The equation of the parabola is $y^2-4ax=0$ ...(1)

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices of the triangle.

Then the area of the triangle is

$\triangle=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$ ...(2)

[To express in terms of y_1, y_2, y_3]

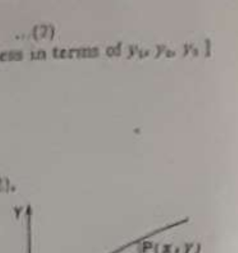
$\because$ (x_1, y_1) lies on the parabola (1)

$\therefore$ $y_1^2-4ax_1=0$ or $x_1=\frac{y_1^2}{4a}$.

Similarly $x_2=\frac{y_2^2}{4a}$, $x_3=\frac{y_3^2}{4a}$.

Substitute these values of x_1, x_2, x_3 in (2).

3. Let A be the vertex and S the focus of the parabola on the x-axis, so that OA$=a$, OS$=a'$.



Let P(x, y) be *any* pt. on the parabola.

From P draw PN $\perp$ on the x-axis.

Then PN^2=4 AS.AN [Art. 122, Cor.]

[But AS=OS−OA$=a'-a$,

AN=ON−OA$=x-a$]

or $y^2=4(a'-a)(x-a)$,

which is the required equation.

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Ex. The equation of the parabola is $y^2=8x$ ...(1)

[Compare with $y^2=4ax$]

Here $4a=8$, $\therefore$ $a=2$.

Let (x_1, y_1) be the required co-ordinates.

Then the focal distance$=2+x_1$ [$a+x_1$ (Art. 123 (*b*))]

$=4$ (Given)

$\therefore$ $x_1=2$.

But (x_1, y_1) lies on the parabola (1), $\therefore$ $y_1^2=8x_1$.

Substitute the value of x_1 found above, and find the values of y_1.

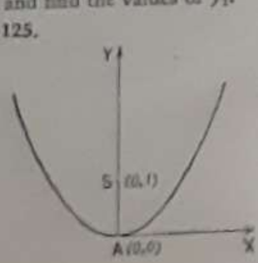
Pages 185—188, Art. 125.

4. The vertex A(0, 0) and the focus S(0, 1) lie on the y-axis. AS=1, *i.e.* $a=1$

$\therefore$ the equation of the (upward) parabola is

$x^2=4.1.y$ [$x^2=4ay$]

or $x^2=4y$.



Page 188.

1. (*i*) The equation of the parabola is $y^2=8x$.

[Compare with $y^2=4ax$]

Here $4a=8$, $\therefore$ $a=2$.

(*ii*) Divide thro'out by 3 (to reduce the equation of the parabola to the form $y^2=4ax$).

(*iii*) (*a*) Use Art. 126 (*b*), N.B.

(*b*) and (*c*). Draw a rough Fig. (Art. 125, Ex. 1), and proceed as in Cor. of that Ex.

Page 189.

4. (*i*) Divide thro'out by 4 to make the coeff. of the square term, namely of $y^2=1$. Proceed as in solved Ex., Art. 127.

Page 192. Art. 128.

2. Proceed as in solved Ex. 2, Art. 128, and get $y=\frac{y_1+y_2}{2}$.

Substitute this value of y in (2) of that Ex., and find the value of x.

Pages 192—193. Art. 129.

6. The equation of the parabola is $y^2=12x$.

[Compare with $y^2=4ax$]

Here $4a=12$, $\therefore$ $a=3$.

$\therefore$ the ends of the latus rectum are $(a, 2a)$, $(a, -2a)$, *i.e.* $(3, 6)$, $(3, -6)$.

8. The equation of the parabola is $y^2=4ax$.

The equation of the normal at (x_1, y_1) is

$$y-y_1=-\frac{y_1}{2a}(x-x_1) \quad ...(1)$$ [To express in terms of y_1]

$\because$ (x_1, y_1) lies on the parabola,

$\therefore$ $y_1^2=4ax_1$ or $x_1=\frac{y_1^2}{4a}$.

Substituting this value of x_1 in (1),

$$y-y_1=-\frac{y_1}{2a}\left(x-\frac{y_1^2}{4a}\right) \quad ...(2)$$

which is the equation of the normal at the pt. whose ordinate is y_1.

Similarly the equation of the normal at the pt. whose ordinate is y_2, is [changing y_1 to y_2 in (2)]

$$y-y_2=-\frac{y_2}{2a}\left(x-\frac{y_2^2}{4a}\right) \quad ...(3)$$

Subtract (3) from (2), and find the value of x. Substitute this value of x in (2), and find the value of y.

Page 194.

2. The equation of the line is $y=mx+c$ ...(1)
and that of the parabola is $y^2=4ax$ ...(2)

Proceed as in Art. 130, and find the equation of the normal in the m-form, namely $y=mx-2am-am^3$ ...(3)

If it is the same as the equation of the given line (1), then comparing constant terms on the R.H.S.'s of (1) and (3),

$c=-2am-am^3$, which is the required condition.

4. The equation of the line is $y=x-6$ ...(1)
and that of the parabola is $y^2=8x$ ...(2)
[Compare with $y^2=4ax$]

Here $4a=8$, $\therefore a=2$.

$\therefore$ the equation of *any* normal to the parabola (2), in the m-form, is

$y=mx-2.2m-2m^3$ ...(3) [$y=mx-2am-am^3$]

Comparing coeffs. of x on the R.H.S.'s of (1) and (3), $m=1$.

Substituting this value of m in (3), it becomes

$y=1.x-4.1-2.1^3=x-6$

$\therefore$ $y=x-6$ is a normal to the parabola (2).

The foot of the normal is $(am^2, -2am)$ [Here $a=2, m=1$]
i.e. $(2.1^2, -2.2.1)$ or $(2, -4)$.

Page 196.

1. (*a*) Proceed as in Art. 132, and get $am^3+m(2a-x_1)+y_1=0$, which is a cubic in m giving three values of m.

$\because$ imaginary roots occur in pairs [From Higher Algebra]

$\therefore$ at least one root of the above cubic is real.

$\therefore$ at least one of the three normals is real.

2. Proceed as in solved Ex. 1 (*ii*), Art. 132.

4. Proceed as in solved Ex. 2, Art. 132, and get

$am^3+m(2a-x_1)+y_1=0$ ...(1)

which is a cubic in m. Let m_1, m_2, m_3 be the roots.

If two normals make complementary angles, say, θ, $90^\circ-\theta$ with the x-axis, then

$m_1m_2=\tan\theta . \tan(90^\circ-\theta)=\tan\theta . \cot\theta=\tan\theta . \frac{1}{\tan\theta}=1$

$m_1m_2=1$ ...(2)

Proceed further as in solved Ex. 2, Art. 132.

Page 198.

1. First find the equation of the normal at (3, 6) to the parabola, and then use the Note given in Ex. 1.

2. (*i*) The equation of the chord is $y=\sqrt{2}\,x-4a\sqrt{2}$.
Proceed as in Ex. 4 (*i*), Art. 130.

3. (*i*) The equation of the line is $y=-x+3$. Proceed as in Ex. 4 (*i*), Art. 130.

Pages 201—202.

2. Proceed as in solved Ex. 1, Art. 88.

6. (*a*) *If a line is equally inclined to the axes, its slope* $m=\pm 1$ [Art. 32, Ex. 3].

7. The equation of the parabola is $y^2=16x$.
[Compare with $y^2=4ax$]

(*i*) Here $4a=16$, $\therefore$ $a=4$; $m=\frac{3}{4}$.

(*ii*) Here $4a=16$, $\therefore$ $a=4$; $m=-\frac{1}{\frac{3}{4}}=-\frac{4}{3}$.

8. The equation of the parabola is $y^2=8x$ ...(1)
[Compare with $y^2=4ax$]

Here $4a=8$, $\therefore$ $a=2$.

The equation of the circle is $x^2+y^2-12x+4=0$ ...(2)

Let m be the slope of a tangent to the parabola (1).

Then its equation is $y=mx+\frac{2}{m}$ ...(3) $\left[y=mx+\frac{a}{m}\right]$

If it touches the circle (2), use the tangent property, and find the values of m.

Substitute these values of m one by one in (3).

Note. *Common tangents.* **Rule to find the equations of the common tangents of two conics.**

(*i*) *Let m be the slope of a tangent to one conic (the simpler the better). Then its equation is ?* ...(*1*)

(*ii*) *If it touches the other conic, then ? Find the values of m.*

(*iii*) *Substitute these values of m one by one in (1).*

10. Proceed as in solved Ex. 2, Art. 88.

Page 203.

Ex. (*iii*) The equation of the parabola is $2y^2=9x$ ...(1)

Dividing thro' out by 2, [To reduce (1) to the form $y^2=4ax$]

$y^2=\frac{9}{2}x$.

Page 206, Art. 138.

Ex. To find the angle between the tangents, use Cor. 2, Art. 69.

Pages 206—207. Art. 139.

1. Proceed as in Art. 139, and get

$$m^2x_1-my_1+a=0 \quad ...(1)$$

which is a quadratic in m. Let m_1, m_2 be the roots.

Then $m_1+m_2=\frac{y_1}{x_1}$, $m_1m_2=\frac{a}{x_1}$ ...(2)

(*i*) $m_1+m_2=k$. Substitute from (2).

(*iii*) $m_1^2+m_2^2=k$ or $(m_1+m_2)^2-2m_1m_2=k$. Substitute from (2).

(*iv*) $\frac{1}{m_1}+\frac{1}{m_2}=k$ or $\frac{m_1+m_2}{m_1m_2}=k$. Substitute from (2).

3. The locus is $y^2-4ax=(a+x)^2\tan^2\alpha$.

(*i*) $(a+x)^2=\frac{y^2-4ax}{\tan^2\alpha}$ ...(A)

If $\alpha=\frac{\pi}{2}$, $\tan\alpha=\tan\frac{\pi}{2}$ which is infinite.

$\therefore$ from (A), the locus is $(a+x)^2=0$ or $a+x=0$.

Page 208.

1. Let the equation of the parabola be $y^2=4ax$.
Then the equation of the directrix is $x=-a$.
If (x_1, y_1)...(1) lies on it, then $x_1=-a$,
$\therefore$ from (1), the pt. is $(-a, y_1)$. Find the equation of the chord of contact of tangents from this pt. to the parabola.

2. Let the equation of the parabola be $y^2=4ax$. The pt. of intersection of two $\perp$ tangents to the parabola lies on the directrix $x=-a$ (Art. 139).

$\therefore$ let this pt. of intersection be $(-a, y_1)$. Find the equation of the chord of contact of tangents from this pt. to the parabola.

3. Proceed as in Ex. 2, Art. 96.

4. The equation of the parabola is $y^2=4ax$.

Let (x_1, y_1)...(1) be the pt. of intersection of the tangents. Then the equation of the chord of contact is $yy_1=2a(x+x_1)$.

If it passes thro' the focus $(a, 0)$, then $0(y_1)=2a(a+x_1)$,
$\therefore x_1=-a$. $\therefore$ from (1), the pt. of intersection is $(-a, y_1)$, which lies on the directrix $x=-a$.

Find the equation of the pair of tangents from $(-a, y_1)$ to the parabola, and show that they are $\perp$. (Transpose, and use coeff. of x^2+coeff. of $y^2=0$.)

Page 209.

2. Let P be the pt. (x_1, y_1).

Page 210. Art. 142.

4. The equation of the parabola is $y^2=4ax$.
Let (x_1, y_1) be the pole of *any* chord.
Then the equation of the chord (*i.e.* polar) is $yy_1=2a(x+x_1)$.
If it is at a distance b from the vertex $(0, 0)$, then ?
Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

6. The equation of the parabola is $y^2=4ax$.
Let (x_1, y_1) be the pole of *any* chord.
Then the equation of the chord (*i.e.* polar) is $yy_1=2a(x+x_1)$
or $2ax-yy_1+2ax_1=0$ ...(1)
If it is the same as the equation of the normal, in the m-form,

$$y=mx-2am-am^3$$

or $mx-y-2am-am^3=0$ ...(2)
then comparing coeffs. in (1) and (2),

$$\frac{2a}{m}=\frac{-y_1}{-1}=\frac{2ax_1}{-2am-am^3}.$$

From the first and second members,

$$\frac{2a}{m}=y_1 \quad ...(3)$$

From the first and third members,

$$1=\frac{x_1}{-2a-am^2} \quad ...(4)$$

Eliminate m from (3) and (4) [by substituting its value $\left(m=\frac{2a}{y_1}\right)$ from (3) in (4)].
Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

7. Let (x_1, y_1) be the pole of *any* line.
Then the equation of the line (*i.e.* polar w.r.t. the parabola $y^2=4bx$) is

$$yy_1=2b(x+x_1)$$

or $2bx-yy_1+2bx_1=0$. [Compare with $lx+my+n=0$]
Here $l=2b$, $m=-y_1$, $n=2bx_1$.
If it touches the parabola $y^2=4ax$,
then $2b(2bx_1)=a(-y_1)^2$. [$l.n=am^2$ (Art. 135, Ex. 10)]
Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

Page 210. Art. 143.

Ex. Proceed as in solved Ex., Art. 100.

Pages 211—212.

6. Let the equation of the parabola be $y^2=4ax$.
Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$, $R(at_3^2, 2at_3)$ be the vertices of the triangle inscribed in the parabola.

Then the pt. of intersection of the tangents at Q, R is

$$P'[at_2t_3,\ a(t_2+t_3)]. \qquad \text{[Ex. 5]}$$

Similarly the pts. of intersection of the tangents at R, P, and at P, Q are

$$Q'[at_3t_1,\ a(t_3+t_1)],\ R'[at_1t_2,\ a(t_1+t_2)].$$

Use the area formula (Second form) (Art. 12), and prove that

$$\triangle PQR = 2\triangle P'Q'R'.$$

7. The equation of the parabola is $y^2=4ax$.

Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ be the pts. on the parabola, so that $\frac{at_1^2}{at_2^2}=n$ or $\frac{t_1^2}{t_2^2}=n$ ...(1)

The co-ordinates of the pt. of intersection of the tangents at P, Q are $x=at_1t_2$ ...(2), $y=a(t_1+t_2)$ ...(3)

To find the locus of the pt. of intersection of the tangents at P, Q, we have to eliminate t_1, t_2 from (1), (2), (3).

Substituting the value of $t_1(=\sqrt{n}\,t_2)$ from (1) in (2), (3),

$x=a\sqrt{n}\,t_2^2$ ...(4), $y=a(\sqrt{n}+1)t_2$ ...(5)

Eliminate t_2 from (4) and (5) [by substituting its value from (5) in (4)].

8. Let the equation of the parabola be $y^2=4ax$.

Find the equation of the circle on the join of $S(a, 0)$ and $P(at^2, 2at)$ as diameter. It meets the tangent at the vertex (y-axis) where, putting $x=0$, it will be found that $(y-at)^2=0$, which has equal roots, each $=at$.

Page 213.

1. The equation of the parabola is $y^2=4ax$.

Let P and Q be the pts. $(at^2, 2at)$, and $\left[\text{changing } t \text{ to } -\frac{1}{t}\right]$

$$\left(\frac{a}{t^2}, -\frac{2a}{t}\right).$$

Then $SP=a+x_1$ [Art. 123(b)]

$$=a+at^2.$$

$$\therefore\ \frac{1}{SP}+\frac{1}{SQ}=\frac{1}{a+at^2}+\frac{1}{a+\frac{a}{t^2}}=?$$

2. We have to prove that $2a$ is the H.M. between SP and SQ, *i.e.* $\frac{1}{2a}=\frac{1}{2}\left[\frac{1}{SP}+\frac{1}{SQ}\right]$. Proceed as in Ex. 1.

3. Let the equation of the parabola be $y^2=4ax$.

Let $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ be the extremities of the focal

chord. Write down the equations of the tangents at P, Q to the parabola.

Prove that the tangents are $\perp$.

Solve the equations of the tangents for x, and prove that $x=-a$.

Pages 214—216.

2. The equation of the parabola is $y^2=4ax$.

Let (x_1, y_1) be the mid-pt. of *any* chord. It will be found that the equation of the chord is

$$yy_1-2ax=y_1^2-2ax_1.$$

If it passes thro' the vertex (0, 0), then ?

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

3. Proceed as in Ex. 2. If the chord passes thro' the focus $(a, 0)$, then ?

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

5. Let the equation of the parabola be $y^2=4ax$ ...(1)

Let (x_1, y_1) be the mid-pt. of *any* chord. It will be found that the equation of the chord is

$$yy_1-2ax=y_1^2-2ax_1$$

or $\frac{yy_1-2ax}{y_1^2-2ax_1}=1$ ...(2) [Constant term on R.H.S. and $=1$]

The equation of the lines joining the origin to the pts. of intersection of the parabola (1) and the chord (2) is [making (1) homogeneous by means of (2)], $y^2=4ax.\frac{yy_1-2ax}{y_1^2-2ax_1}$.

If these lines are $\perp$, then ? [Transpose, and use coeff. of x^2+coeff. of $y^2=0$.]

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

6. The equation of the parabola is $y^2=4ax$.

Let (x_1, y_1) be the mid-pt. of *any* chord. It will be found that the equation of the chord is

$$yy_1-2ax=y_1^2-2ax_1$$

or $2ax-yy_1+(y_1^2-2ax_1)=0$ ...(1)

If it is the same as the equation of the normal, in the m-form, namely

$$y=mx-2am-am^3$$

or $mx-y-2am-am^3=0$ ...(2)

then comparing coeffs. in (1) and (2),

$$\frac{2a}{m}=\frac{-y_1}{-1}=\frac{y_1^2-2ax_1}{-2am-am^3}.$$

From the first and second members,

$$\frac{2a}{m}=y_1 \quad ...(3)$$

From the first and third members,

$$2a=\frac{y_1^2-2ax_1}{-2a-am^2} \quad ...(4)$$

Eliminate m from (3) and (4) [by substituting its value $\left(m=\frac{2a}{y_1}\right)$ from (3) in (4)].

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

7. The equation of the parabola is $y^2=8x$ or $y^2-8x=0$ ...(1)

[Form : S=0]

Find the equation of the chord whose mid-pt. is (1, 2).
Find its pole w.r.t. the parabola.

Page 219. Art. 148 (*e*).

2. The equation of the parabola is $y^2=6x$.

[Compare with $y^2=4ax$]

Here $4a=6$, $\therefore a=\frac{3}{2}$, $m=3$. Use $y=\frac{2a}{m}$.

MISCELLANEOUS EXAMPLES ON CHAPTER VII

Pages 219—223.

5. (*i*) The equation of the parabola is $4y^2+12x-20y+67=0$.

Divide thro' out by 4 (to make coeff. of square term, namely of $y^2=1$).

Proceed as in solved Ex., Art. 127.

It will be found that $(y-\frac{5}{2})^2=-3(x+\frac{7}{2})$ ...(1)

Transferring the origin to $(-\frac{7}{2}, \frac{5}{2})$ by putting

$$x=x'-\tfrac{7}{2},\ y=y'+\tfrac{5}{2} \quad ...(2)$$

(1) becomes $y'^2=-3x'$ ...(3)

It is a left-handed parabola, whose axis runs in the $-$ve direction of the *new* x-axis.

[Compare (3) with $y'^2=-4ax'$]

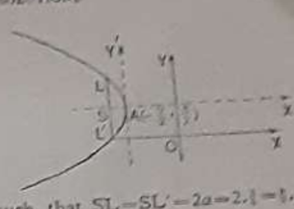
Here $4a=3$, $\therefore a=\frac{3}{4}$.

The vertex A is the *new* origin.

The focus S is on the $-$ve side of the *new* x-axis, and $AS=a=\frac{3}{4}$.

The latus rectum LSL$'$ is $\perp$ to AS, and $SL=SL'=2a=2(\frac{3}{4})=\frac{3}{2}$.

Hence the parabola as shown in the Fig, on the next page.



[**How to draw the parabola.**

Draw two $\perp$ axes OX, OY, and plot the pt. $A(-\frac{7}{2}, \frac{5}{2})$. Then A is the vertex of the parabola. Thro' A draw the new axes AX$'$, AY$'$ $\parallel$ to OX, OY respectively. On AX$'$ take a pt. S to the *left* of A such that $AS=a=\frac{3}{4}$. Then S is the focus.

Thro' S draw LSL$'$ $\perp$ to AS such that $SL=SL'=2a=2.\frac{3}{4}=\frac{3}{2}$. Then L, L$'$ are the extremities of the latus rectum.

Thro' L, A, L$'$ draw a parabola touching AY$'$ at A.]

6 The equation of the parabola is $y^2=4ax$ ...(1)

Let AP be the chord passing thro' the vertex A, so that $\angle XAP=\theta$.

Let $AP=r$. From P draw PN $\perp$ on AX. Then $AN=r\cos\theta$, $NP=r\sin\theta$. $\therefore$ P is $(r\cos\theta, r\sin\theta)$.

$\because$ P lies on the parabola (1), $\therefore$? Find the value of r.

9. Use the Rule of Art. 80.

10. The equations of the parabolas are $y^2=4ax$ ...(1)

$x^2=4by$ ...(2)

Proceed as in Ex. 8 (*ii*), and find the pts. of intersection (0, 0) and $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$.

The equations of the tangents to the parabolas (1) and (2) at $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$ are

$$y(4a^{\frac{2}{3}}b^{\frac{1}{3}})=2a(x+4a^{\frac{1}{3}}b^{\frac{2}{3}}), \qquad [yy_1=2a(x+x_1)]$$

and $$x(4a^{\frac{1}{3}}b^{\frac{2}{3}})=2b(y+4a^{\frac{2}{3}}b^{\frac{1}{3}})$$

[$xx_1=2b(y+y_1)$ (Rule of Art. 81, Note)]

The slopes of the tangents are

$$\frac{2a}{4a^{\frac{2}{3}}b^{\frac{1}{3}}}=\frac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}},\ \frac{4a^{\frac{1}{3}}b^{\frac{2}{3}}}{2b}=\frac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}};$$

find the angle between the tangents.

Note. The equations of the tangents to the parabolas (1) and (2) at (0, 0) are

$y(0)=2a(x+0)$ [$yy_1=2a(x+x_1)$]

and $x(0)=2b(y+0)$ [$xx_1=2b(y+y_1)$ (Rule of Art. 81, Note)]

or $x=0$ (*i.e.* y-axis) and $y=0$ (*i.e.* x-axis), which are at rt. $\angle$s.

$\therefore$ at the origin, the parabolas intersect at rt. $\angle$s.

11. The equation of the line is $2x-2y+1=0$

or $$y=x+\tfrac{1}{2} \quad \ldots(1)$$

[Form : $y=mx+c$]

and that of the parabola is $y^2=4x \quad \ldots(2)$

Proceed as in Art. 134.

12. Caution. See 'Caution' in Ex. 8. When the values of x (for the points of intersection) have been found, substitute them one by one *in the equation of the normal and not in the equation of the parabola.*

The same caution should be observed when finding the points of intersection of a line and a conic. See step (*ii*) of the Rule given in Note 1, Art. 86.

17. The equations of the parabolas are
$$y^2=4ax \quad \ldots(1)$$
$$x^2=4by \quad \ldots(2)$$

Let m be the slope of a tangent to the parabola (1).

Then its equation is $y=mx+\dfrac{a}{m} \quad \ldots\ldots(3)$

Substituting the value of y from (3) in (2),
$$x^2=4b\left(mx+\frac{a}{m}\right) \text{ or } x^2-4bmx-\frac{4ab}{m}=0 \quad \ldots(4)$$

If the tangent (3) touches the parabola (2) also, the quadratic (4) has equal roots,
$$\therefore 16\,b^2m^2-4\left(-\frac{4ab}{m}\right)=0 \qquad [b^2-4ac=0]$$
$$\therefore m^3=-\frac{a}{b} \text{ or } m=-\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}.$$

Substitute this value of m in (3).

18. The equations of the curves are
$$y^2=4ax \quad \ldots(1)$$
$$x^2+y^2+2ax=0 \quad \ldots(2)$$

Let m be the slope of a tangent to the parabola (1).

Then its equation is $y=mx+\dfrac{a}{m} \quad \ldots(3)$

If it touches the circle (2), use the tangent property, and find the values of m. Substitute these values of m one by one in (3).

19. The equation of the parabola is $y^2=4x \quad \ldots(1)$

[Compare with $y^2=4ax$]

Here $4a=4, \therefore a=1$.

Let m be the slope of a tangent to the parabola (1).

Then its equation is $y=mx+\dfrac{1}{m} \quad \ldots(2) \left[y=mx+\dfrac{a}{m}\right]$

If it makes an angle of 45° with the line $y=2x+3$ (slope $=2$),

then $$\tan 45^\circ=\pm\frac{m-2}{1+m\,(2)}.$$

Find the values of m, and substitute them one by one in (2).

The pt. of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$, *i.e.* $\left(\dfrac{1}{m^2}, \dfrac{2}{m}\right) \quad \ldots(3)$

Substitute the values of m (found above) one by one in (3).

21. The equation of the parabola is $y^2=4ax \quad \ldots(1)$

The equation of the chord of contact of tangents from (x_1, y_1) to the parabola (1) is
$$yy_1=2a(x+x_1) \quad \ldots(2)$$

[To find the length of the intercept made by the parabola (1) on the line (2).]

Substituting the value of y from (2) in (1),
$$\left[\frac{2a(x+x_1)}{y_1}\right]^2=4ax \text{ or } 4a^2(x^2+2xx_1+x_1^2)=4axy_1^2 \qquad [\text{Cancel } 4a]$$

or $$ax^2+x(2ax_1-y_1^2)+ax_1^2=0 \quad \ldots(3)$$

which is a quadratic in x, giving two values of x, say, x', x''.

Substituting these values of x one by one in (2), we get the corresponding values of y, namely y', y'', such that
$$y'y_1=2a(x'+x_1) \quad \ldots(4)$$
$$y''y_1=2a(x''+x_1) \quad \ldots(5)$$

Now the pts. of intersection are $(x', y'), (x'', y'')$.

$\therefore$ length of the intercept $=\sqrt{(x''-x')^2+(y''-y')^2}$

[But subtracting (4) from (5),
$y_1(y''-y')=2a(x''-x')$]

$$=\sqrt{(x''-x')^2+\frac{4a^2}{y_1^2}(x''-x')^2} = (x''-x')\frac{\sqrt{y_1^2+4a^2}}{y_1} \quad \ldots(6)$$

But from (3), $$x''+x' = -\frac{2ax_1-y_1^2}{a}, \text{ [Sum of the roots]}$$

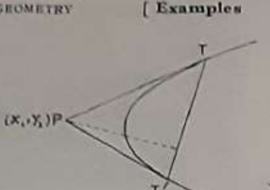
and $$x''x' = \frac{ax_1^2}{a}=x_1^2. \text{ [Product of the roots]}$$

$$\therefore x''-x'=\sqrt{(x''+x')^2-4x''x'}=\sqrt{\frac{(2ax_1-y_1^2)^2}{a^2}-4x_1^2}$$
$$=\frac{\sqrt{-4ax_1y_1^2+y_1^4}}{a}=\frac{y_1\sqrt{y_1^2-4ax_1}}{a}.$$

Substitute this value of $x''-x'$ in (6).

22. The equation of the parabola is $y^2=4ax$ ...(1)

Let P be the pt. (x_1, y_1), and PT, PT' the tangents from P to the parabola, T, T' being the pts. of contact.



Then the equation of TT', the chord of contact of tangents from P to the parabola (1), is

$$yy_1=2a(x+x_1) \quad ...(2)$$

Proceed as in Ex. 21, and find the length of the chord TT'.

Now $\triangle$ PTT' $=\frac{1}{2}$TT'. ($\perp$ from P on TT') = ?

24. Let (x_1, y_1) be *any* pt.

25. The equation of the parabola is $y^2=4ax$ ...(1)

Let the tangents at the extremities of a chord meet at (x_1, y_1).

Then the equation of the chord of contact is $yy_1=2a(x+x_1)$

or $\dfrac{yy_1-2ax}{2ax_1}=1$ ...(2) [Constant term on R.H.S. and $=1$]

The equation of the lines joining the origin to the pts. of intersection of the parabola (1) and the chord (2) is [making (1) homogeneous by means of (2)]

$$y^2=4ax\left(\frac{yy_1-2ax}{2ax_1}\right).$$

If these lines are $\perp$, then ? [Transpose, and use coeff. of x^2 + coeff. of $y^2=0$.]

It will be found that $x_1+4a=0$,

$\therefore$ (x_1, y_1) lies on the line $x+4a=0$.

26. The equations of the parabolas are $y^2=-4bx$ ...(1)

$y^2=4ax$ ...(2)

Let (x_1, y_1) be the mid-pt. of *any* chord PQ of the parabola (2). Then it will be found that the equation of PQ is

$$yy_1-2ax=y_1^2-2ax_1$$

or $$2ax-yy_1+(y_1^2-2ax_1)=0.$$

[Compare with $lx+my+n=0$, here $l=2a$, $m=-y_1$, $n=y_1^2-2ax_1$]

If it touches the parabola (1)

[Compare (1) with $y^2=4ax$, here "a" $=-b$]

then $2a.(y_1^2-2ax_1)=-b(-y_1)^2$. [$l.n=am^2$ (Art. 135, Ex. 10)]

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

27. Let the equation of the parabola be $y^2=4ax$.

Let P be the pt. (x_1, y_1). The equation of the normal at P to the parabola is $y-y_1=-\dfrac{y_1}{2a}(x-x_1)$ ...(1)

It meets the x-axis where, putting $y=0$ in (1), it will be found that $x=2a+x_1$. $\therefore$ G is $(2a+x_1, 0)$.

It will be found by the distance formula that

$\text{PG}=\sqrt{4a^2+y_1^2}=\sqrt{4a^2+4ax_1}$

[$\because$ P being on the parabola, $y_1^2=4ax_1$]

$=\sqrt{4a(a+x_1)}=\sqrt{4a.\text{SP}}$

[$\because$ focal distance SP $=a+x_1$ (Art. 123(b))]

$\therefore$ PG is a geometric mean between $4a$ (*i.e.* latus rectum) and SP.

28. (*a*) Prove that the locus of the foot of the $\perp$ from the focus on any tangent to a parabola is the tangent at the vertex. [Proceed as in Art. 148 (c).]

(*b*) The equation of *any* tangent to the parabola, in the m-form, is

$$y=mx+\frac{a}{m} \quad ...(1)$$

$\therefore$ SY $=\perp$ from the focus S$(a, 0)$ on the tangent (1), which will be found

$$=\frac{a}{m}\sqrt{m^2+1}.$$

$$\therefore \text{SY}^2=\frac{a^2}{m^2}(m^2+1).$$

P, the pt. of contact of the tangent (1), is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

$$\therefore \text{AS.SP}=a(a+x_1)=a\left(a+\frac{a}{m^2}\right)$$

$$=\frac{a^2}{m^2}(m^2+1)$$

$\therefore$ SY^2=AS.SP.

29. Let the equation of the parabola be $y^2=4ax$. Let P$(at^2, 2at)$ be the pt. of contact. It will be found that the equation of the tangent at P is $ty=x+at^2$ ...(1)

It meets the directrix $x=-a$ where [putting $x=-a$ in (1)]

$y=\dfrac{at^2-a}{t}$, *i.e.* at K$\left(-a, \dfrac{at^2-a}{t}\right)$; S is $(a, 0)$.

Find the slopes of SP and SK, and show that SP is $\perp$ to SK.

30. The equation of the parabola is $y^2=4ax$.

The equation of the normal at $(at_1^2, 2at_1)$ is

$$y-2at_1=-\frac{2at_1}{2a}(x-at_1^2) \qquad \left[y-y_1=-\frac{y_1}{2a}(x-x_1)\right]$$

or $$y-2at_1=-t_1(x-at_1^2).$$

If it passes thro' $(at_2^2, 2at_2)$, then ?

Cancel the common factor (t_2-t_1) on both sides.

31. The equation of the parabola is $y^2=4ax$.

Let P be the pt. $(at^2, 2at)$ on the parabola. From P draw PN $\perp$ on the x-axis, and produce it to meet the parabola in P'.

Then P' is $(at^2, -2at)$. It will be found that the equation of the normal at P is

$$y-2at=-tx+at^3 \quad ...(1)$$

and the equation of the line $\parallel$ to the x-axis, and passing thro' P' is

$$y=-2at \quad ...(2)$$

Eliminate t from (1) and (2) [by substituting its value $\left(t=-\frac{y}{2a}\right)$ from (2) in (1)]. Cancel the common factor y.

32. The equation of the parabola is $y^2=4ax$.

Let P be the pt. $(at^2, 2at)$. Find the equation of the normal at P.

It meets the x-axis where, putting $y=0$, $x=$? It will be found that G is $(2a+at^2, 0)$.

$\therefore$ the co-ordinates of the mid-pt. of PG are

$$x=at^2+a, \; y=at.$$

Eliminate t.

33. Let the equation of the parabola be $y^2=4ax$.

Let O be the pt. $(k, 0)$, and P, Q the pts. $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

The equation of the chord PQ is $\frac{t_1+t_2}{2}y=x+at_1t_2$.

It passes thro' O$(k, 0)$, $\therefore 0=k+at_1t_2$.

34. Let the equation of the parabola be $y^2=4ax$.

Let P, Q, R be the pts. $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$, so that $(2at_2)^2=2at_1 \cdot 2at_3$ or $t_2^2=t_1t_3 \quad ...(1)$

The pt. of intersection of the tangents at P, R is T $[at_1t_3, a(t_1+t_3)]$ i.e. from (1), T $[at_2^2, a(t_1+t_3)]$.

Find the equation of the line $\parallel$ to the y-axis, and passing thro' Q $\quad ...(2)$

Show that T lies on the line (2).

35. Let the equation of the parabola be $y^2=4ax$.

(i) Let P$(at^2, 2at)$, and $\left(\text{changing } t \text{ to } -\frac{1}{t}\right)$

Q $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ be the extremities of the focal chord.

36. The equation of the parabola is $y^2=4ax$.

Let P $(at^2, 2at)$, and $\left(\text{changing } t \text{ to } -\frac{1}{t}\right)$

Q $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ be the extremities of the focal chord.

Then $\text{PQ}=\text{SP}+\text{SQ}=a+at^2+a+\frac{a}{t^2}$ [$\because$ SP $=a+x_1$ (Art. 121 (i))]

$$=a\left(t+\frac{1}{t}\right)^2.$$

Now $\tan \alpha=$ slope of PQ, which will be found

$$=\frac{2}{t-\frac{1}{t}}$$

or $$t-\frac{1}{t}=2\cot\alpha \quad ...(1)$$

Now $\text{PQ}=a\left(t+\frac{1}{t}\right)^2=a\left[\left(t-\frac{1}{t}\right)^2+4\right]$.

Substitute from (1).

37. Let the equation of the parabola be $y^2=4ax$.

Let P$(at^2, 2at)$, and $\left(\text{changing } t \text{ to } -\frac{1}{t}\right)$

Q $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ be the extremities of any focal chord.

Find the equation of the circle on PQ as diameter. It meets the directrix $x=-a$ where, putting $x=-a$, it will be found that

$$y^2-2ay\left(t-\frac{1}{t}\right)-4a^2+a^2(1+t^2)\left(1+\frac{1}{t^2}\right)=0$$

or $$y^2-2ay\left(t-\frac{1}{t}\right)+a^2\left[t^2+\frac{1}{t^2}-2\right]=0 \text{ or } \left[y-a\left(t-\frac{1}{t}\right)\right]^2=0,$$

which has equal roots, each $=a\left(t-\frac{1}{t}\right)$.

CHAPTER VIII

Page 228.

3. (ii) Let the required equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ...(1)$$

$\because$ it passes thro' $(-3, 1)$, $(2, -2)$,

$$\therefore \frac{9}{a^2}+\frac{1}{b^2}=1 \quad ...(2)$$
$$\frac{4}{a^2}+\frac{4}{b^2}=1 \quad ...(3)$$

From (2) and (3), find the values of $\frac{1}{a^2}$, $\frac{1}{b^2}$, and substitute these values in (1).

For eccentricity, use $b^2=a^2(1-e^2)$.

Pages 234—235.

1. $2\frac{b^2}{a}=4$ ...(1), $SA=CA-CS=a-ae=\frac{3}{2}$ (Given) ...(2)

From (1), $\frac{2a^2(1-e^2)}{a}=4$ or $a(1-e^2)=2$ ...(3)

From (2), $a(1-e)=\frac{3}{2}$ ...(4)

Divide (3) by (4).

5. (*i*) $2\frac{b^2}{a}=\frac{1}{2}.2b=b$, $\therefore 2\frac{b}{a}=1$ [But $b^2=a^2(1-e^2)$ $\therefore b=a\sqrt{1-e^2}$]

or $2\frac{a\sqrt{1-e^2}}{a}=1$. Cancel a, and square.

6. It will be found that $e=\frac{-1\pm\sqrt{5}}{2}$.

$\therefore$ (1) $e=\frac{\sqrt{5}-1}{2}$.

(2) $e=-\frac{\sqrt{5}+1}{2}$ which, being numerically >1, is impossible. ($\because$ eccentricity of an ellipse is <1)

Page 238.

2. (*i*) Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The extremities of the major axis are $A(-a, 0)$, $A'(a, 0)$.

(*ii*) The extremities of the minor axis are $B(0, b)$, $B'(0, -b)$.

3. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

It will be found that the equation of the tangent at $L\left(-ae, \frac{b^2}{a}\right)$ to the ellipse is

$$ex-y+a=0 \quad ...(1)$$

Substituting the co-ordinates of $Z\left(-\frac{a}{e}, 0\right)$ in (1), we get

$$e\left(-\frac{a}{e}\right)-0+a=0 \text{ or } 0=0, \text{ which is true.}$$

$\therefore$ the tangent at L passes thro' Z.

Similarly the tangent at L′ passes thro' Z, and the tangents at the ends of the latus rectum thro' S′ pass thro' Z′.

Page 239.

2. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Find the equation of the normal at $\left(ae, \frac{b^2}{a}\right)$, and in the equation obtained, put $b^2=a^2(1-e^2)$.

Page 241.

1. The equation of the line is $y=x+1$ ...(1)

and that of the ellipse is $\frac{1}{4}x^2+y^2=1$ ...(2)

Find the pts. of intersection of the line (1) and the ellipse (2).

(*a*) Now length of the chord
= distance between the pts. of intersection.

Page 243.

1. The equation of the line is $y=mx+c$ ...(1)

and that of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(2)

Proceed as in Ex. 1, Art. 88.

The equation of the circle is $x^2+y^2=r^2$ or $\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$.

Comparing with (2), here $a^2=r^2$, $b^2=r^2$.

$\therefore$ in the condition of tangency for the ellipse, put $a^2=r^2$, $b^2=r^2$.

4. Proceed as in Art. 159.

5. Use the method of pt. of contact. (Proceed as in solved Ex. 2, Art. 88.)

6. Use the method of pt. of contact. (Proceed as in solved Ex. 2, Art. 88.)

8. See Hint to Ex. 6, Art. 56.

Page 245.

2. The equation of the ellipse is $5x^2+7y^2=140$ ...(1)

Dividing thro' out by 140, $\left[\text{to reduce (1) to the form } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right]$

$$\frac{x^2}{28}+\frac{y^2}{20}=1.$$

Page 248.

2. Proceed as in Art. 163, and get

$$m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-b^2)=0 \quad ...(1)$$

which is a quadratic in m. Let m_1, m_2 be the roots.

Then $m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$, $m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2}$ ...(2)

(i) $m_1+m_2=k$. Substitute from (2).

(ii) $m_1m_2=k$. Substitute from (2).

(iii) $\frac{1}{m_1}+\frac{1}{m_2}=k^2$ or $\frac{m_1+m_2}{m_1m_2}=k^2$. Substitute from (2).

(iv) $\theta_1+\theta_2=2\alpha$, $\therefore \tan(\theta_1+\theta_2)=\tan 2\alpha$

or $\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}=\tan 2\alpha$ or $\frac{m_1+m_2}{1-m_1m_2}=\tan 2\alpha$. Substitute from (2).

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3. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

The equation of the chord of contact of tangents from (x_1, y_1) to the ellipse is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(2)

The equation of the lines joining the origin to the pts. of intersection of the ellipse (1) and the chord of contact (2) is [making (1) homogeneous by means of (2)],

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2.$$

If these lines are $\perp$, then ? [Transpose, and use coeff. of x^2 + coeff. of $y^2=0$.]

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

Page 250, Art. 166.

3. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Let the focus of the ellipse be $(-ae, 0)$. Then the equation of the corresponding directrix is $x=-\frac{a}{e}$ or $ex+a=0$ ...(2)

Show that the pole of the directrix (2) w.r.t. the ellipse (1) is the focus $(-ae, 0)$.

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1. Proceed as in solved Ex., Art. 100.
2. The equation of the first line is $lx+my+n=0$ ...(1)

and that of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

and that of the second line is $l'x+m'y+n'=0$ ...(2)

It will be found that the pole of the line (1) w.r.t. the ellipse is $\left(-\frac{l}{n}a^2, -\frac{m}{n}b^2\right)$.

If it lies on the line (2), it will be found that

$$a^2ll'+b^2mm'=nn'.$$

The symmetry* of this result shows that it is also the condition that the pole of the line (2) should lie on the line (1).

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Ex. (b) Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Then the equation of its auxiliary circle is $x^2+y^2=a^2$ ...(2)

Let the pt. be (x_1, y_1).

The equations of the polars of (x_1, y_1) w.r.t. the ellipse (1) and the auxiliary circle (2) are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad ...(3)$$

and $$xx_1+yy_1=a^2 \quad ...(4)$$

Solve (3) and (4) for y,

and prove that $y=0$.

$\therefore$ the polars meet on the major axis.

Pages 254—256.

4. (b) The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

It will be found that the equation of the tangent to the ellipse at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

or $$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}-1=0 \quad ...(1)$$

If it is the same as the equation of the given line

$$lx+my+n=0 \quad ...(2)$$

then comparing coeffs. in (1) and (2),

$$\frac{\frac{\cos\theta}{a}}{l}=\frac{\frac{\sin\theta}{b}}{m}=\frac{-1}{n} \text{ or } \frac{\cos\theta}{al}=\frac{\sin\theta}{bm}=-\frac{1}{n},$$

$$\therefore \cos\theta=-\frac{al}{n} \quad ...(3), \quad \sin\theta=-\frac{bm}{n} \quad ...(4)$$

---

*Explanation. 'The symmetry of this result shows' means that on changing l to l', l' to l; m to m', m' to m; n to n', n' to n, the result remains unchanged.

For $a^2ll'+b^2mm'=nn'$ becomes $a^2l'l+b^2m'm=n'n$ or $a^2ll'+b^2mm'=nn'$.

Eliminating θ from (3) and (4) [by squaring (3) and (4), and adding],

$$1=\frac{a^2l^2}{p^2}+\frac{b^2m^2}{p^2}. \quad (\because \cos^2\theta+\sin^2\theta=1)$$

5. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Write down the equations of the tangents to the ellipse at $(a\cos\theta_1, b\sin\theta_1)$, $(a\cos\theta_2, b\sin\theta_2)$, and solve them for $\frac{x}{a}$, $\frac{y}{b}$ by cross-multiplication as in Misc. Ex. 15 (ii), Chap. III.

6. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let θ_1, θ_2 be the eccentric angles of two pts. on the ellipse, so that $\theta_1+\theta_2=\text{constant}=2\alpha$(say).

Then the co-ordinates of the pt. of intersection of the tangents at the pts. are

$$x=a\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}=a\frac{\cos\alpha}{\cos\frac{\theta_1-\theta_2}{2}} \quad ...(1)$$

$$y=b\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}=b\frac{\sin\alpha}{\cos\frac{\theta_1-\theta_2}{2}} \quad ...(2)$$

Eliminate θ_1, θ_2 from (1) and (2) [by dividing (2) by (1)].

7. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let θ_1, θ_2 be the eccentric angles of two pts. on the ellipse, so that $\theta_1-\theta_2=\frac{\pi}{2}$.

Then the co-ordinates of the pt. of intersection of the tangents at the pts. are

$$x=a\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}=a\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\pi}{4}}=\sqrt{2}\,a\cos\frac{\theta_1+\theta_2}{2} \quad ...(1)$$

$$y=b\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}=b\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\pi}{4}}=\sqrt{2}\,b\sin\frac{\theta_1+\theta_2}{2} \quad ...(2)$$

[To eliminate θ_1, θ_2 from (1) and (2).]

From (1), $\frac{x}{a}=\sqrt{2}\cos\frac{\theta_1+\theta_2}{2}$ ...(3)

and from (2), $\frac{y}{b}=\sqrt{2}\sin\frac{\theta_1+\theta_2}{2}$ ...(4)

Eliminate θ_1, θ_2 from (3) and (4) [by squaring (3) and (4), and adding].

8. Proceed as in solved Ex. 2, Art. 168.

9. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let (x_1, y_1) be the pole of any chord.

Then the equation of the chord (i.e. polar) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad ...(1)$$

If it is the same as the equation of the normal at $(a\cos\theta, b\sin\theta)$, namely

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad ...(2),$$

then comparing coeffs. in (1) and (2), $\cos\theta=?$ ...(3), $\sin\theta=?$ ...(4).

Eliminate θ from (3) and (4) [by squaring (3) and (4), and adding].

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

Page 237.

1. (i) Proceed as in Art. 170.
   (ii) Proceed as in Art. 170.

2. (i) The equation of the chord thro' the pts. whose eccentric angles are θ, ϕ is $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$.

If it passes thro' the focus $(ae, 0)$, then ?

(ii) Proceed as in Part (i), and get

$$e\cos\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2} \quad ...(1)$$

Now $b^2=a^2(1-e^2)$ or $\frac{b^2}{a^2}=1-e^2$ or $e^2=1-\frac{b^2}{a^2}=\frac{a^2-b^2}{a^2}$ ...(2)

Substitute the value of e from (2) in (1). Square.

3. Proceed as in Ex. 2 (i), and get

$$e\cos\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2} \text{ or } \frac{e}{1}=\frac{\cos\frac{\theta-\phi}{2}}{\cos\frac{\theta+\phi}{2}}$$

Apply componendo and dividendo.* $\frac{1+e}{1-e}=\frac{\cos\frac{\theta+\phi}{2}+\cos\frac{\theta-\phi}{2}}{\cos\frac{\theta+\phi}{2}-\cos\frac{\theta-\phi}{2}}=?$

(Use "C, D" formulae of Elementary Trigonometry.)

4. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the chord joining two pts. whose eccentric angles are α, β is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

If it passes thro' $(d, 0)$, then ? It will be found that

$$\frac{d}{a}\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2} \text{ or } \frac{d}{a}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}.$$

Apply componendo and dividendo. $\frac{d+a}{d-a}=\frac{\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}}=?$

(Use "C, D" formulae of Elementary Trigonometry.)

Pages 258—259.

2. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let (x_1, y_1) be the mid-pt. of *any* chord of the ellipse. Then it will be found that the equation of the chord is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

If it passes thro' A′ $(a, 0)$, the +ve end of the major axis, then ?
Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

4. Proceed as in Ex. 2.

6. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let (x_1, y_1) be the mid-pt. of *any* chord of the ellipse. Then it will be found that the equation of the chord is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \quad ...(1)$$

---

*Why this step. In order to prove that $\tan\frac{\theta}{2}\tan\frac{\phi}{2}+\frac{1-e}{1+e}=0$, we have to prove that $\frac{1-e}{1+e}=-\tan\frac{\theta}{2}\tan\frac{\phi}{2}$.

∴ we apply componendo and dividendo.

If it is the same as the equation of the normal at $(a\cos\theta, b\sin\theta)$, namely

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad ...(2)$$

then comparing coeffs. in (1) and (2), $\cos\theta=?$ ...(3) $\sin\theta=?$ ...(4)
Eliminate θ from (3) and (4) [by squaring (3) and (4), and adding].
Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

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1. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P be the pt. (x_1, y_1). It will be found that the equation of the tangent to the ellipse at $L\left(-ae, \frac{b^2}{a}\right)$ is

$$ex-y+a=0 \quad ...(1)$$

The equation of the line ∥ to the y-axis, and passing thro' $P(x_1, y_1)$ is

$$x=x_1 \quad ...(2)$$

It meets the tangent (1) [where putting $x=x_1$ in (1)],

$$ex_1-y+a=0$$

or $y=a+ex_1$, i.e. $MQ=a+ex_1=SP$ [Art. 123 (*b*)].

2. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $P(x_1, y_1)$ be the pt. of contact. Then the equation of the tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(1)

p=length of the ⊥ from the centre $(0, 0)$ on the tangent (1)

$$=\frac{1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}},$$

p'=length of the ⊥ from the focus $(-ae, 0)$ on the tangent (1) (which will be found)

$$=\frac{\frac{a+ex_1}{a}}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}}=\frac{r}{a}\cdot p. \qquad [\because a+ex_1=SP=r]$$

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Ex. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

The pt. is $(3, \frac{12}{5})$.

Length of the subtangent $=\frac{a^2-x_1^2}{x_1}$ [As in Art. 172 (*b*), Cor.]
(Here $x_1=3$)

$$=\frac{a^2-9}{3}=\frac{16}{3} \text{ (Given)}, \therefore a^2=?$$

Also (3, [illegible]) lies on the ellipse (1), ∴ ?
Substituting the value of a^2 found above, $b^2=?$
Now $b^2=a^2(1-e^2)$, ∴ ? ∴ $e=?$

Pages 264—265.

1.* Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let P be the pt. (x_1, y_1). Then $CG=e^2x_1$. [As in Art. 172 (d)]
Now $SG=CS+CG=ae+e^2x_1$ [$\because CG=e^2x_1$]
$=e(a+ex_1)=e.SP$. [$\because SP=a+ex_1$ (Art. 172 (a))]

2. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let P be the pt. (x_1, y_1).
The equation of the normal at P is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2 \quad ...(1)$$

It meets the x-axis where, putting $y=0$ in (1),

$x=\frac{a^2-b^2}{a^2}x_1$, ∴ M is $\left(\frac{a^2-b^2}{a^2}x_1, 0\right)$

∴ PM = ? (Use distance formula.)
The normal meets the y-axis where, putting $x=0$ in (1),

$y=\frac{b^2-a^2}{b^2}y_1$, ∴ N is $\left(0, \frac{b^2-a^2}{b^2}y_1\right)$

∴ PN = ? (Use distance formula.)

$\therefore \frac{PM}{PN}=?$

Page 266.

Ex. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of *any* tangent to the ellipse, in the m-form, is

$$y=mx+\sqrt{a^2m^2+b^2}.$$

The two pts. on the minor axis, each distant $\sqrt{a^2-b^2}$ from the [illegible]re, are $(0, \sqrt{a^2-b^2})$, $(0, -\sqrt{a^2-b^2})$.

Page 267.

[illegible]. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

[illegible]et (x_1, y_1) be one extremity of a diameter. Then the other [illegible]ity is $(-x_1, -y_1)$.

[$\because$ the mid-pt. of the diameter is the centre (0, 0)]

2. The equation of the ellipse is $2x^2+3y^2=4$

or, dividing thro' out by 4, $\frac{x^2}{2}+\frac{3}{4}y^2=1$

or $\frac{x^2}{2}+\frac{y^2}{\frac{4}{3}}=1$. [R.H.S. = 1, and coeffs. of x^2, y^2 as denoms.]

[Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$]

Use $y=-\frac{b^2}{a^2m}x$.

Page 268.

3. (i) The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the diameter is $x+y=0$, ∴ $m=-1$.
∴ the equation of the diameter conjugate to the above diameter

is $y=-\frac{b^2}{a^2(-1)}x$. [$y=-\frac{b^2}{a^2m}x$]

Page 270.

Ex. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Let P($a\cos\theta$, $b\sin\theta$) be an extremity of a diameter.
Then an extremity of the conjugate diameter is
D($-a\sin\theta$, $b\cos\theta$).
The equation of the tangent at P to the ellipse (1) is ?
∴ the slope of the tangent is ? The slope of CD is ?

MISCELLANEOUS EXAMPLES ON CHAPTER VIII

Pages 271—274.

4. (c) Proceed as in solved Ex., Art. 154. It will be found that the equation of the ellipse is

$2(x-[illegible])^2+y^2=[illegible]$ ...(1)

Transferring the origin to ([illegible], 0), by putting

$x=x'+[illegible]$, $y=y'+0$ ...(2)

it will be found that (1) becomes

$\frac{x'^2}{[illegible]}+\frac{y'^2}{[illegible]}=1$ ...(3)

Here denominator of y'^2 > denominator of x'^2.

(Second standard form)

From (3),
(greater denominator) $a^2=[illegible]$,
(lesser denominator) $b^2=[illegible]$.

The extremities of the minor axis are $x' = \pm b, y' = 0$,
i.e. $x' = \pm$ [illegible], $y' = 0$ or from (2) ?

5. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

and that of the circle is $x^2 + y^2 = ab$ ...(2)

Solving (1) and (2), it will be found that $x^2 = \frac{a^2 b}{a+b}$, $y^2 = \frac{ab^2}{a+b}$.

$\therefore$ a pt. of intersection is $\left(\sqrt{\frac{a^2 b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$.

It will be found that the slope of the tangent at this pt. to the ellipse (1) is $m_1 = -\frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}$,

and that of the tangent to the circle (2) is $m_2 = -\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$.

$\therefore$ if θ is the angle between the tangents, then

$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$. Substitute the values of m_1, m_2 found above.

6. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let (x_1, y_1) be the required co-ordinates of the pt.
The equation of the tangent at (x_1, y_1) to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

$\because$ it makes equal angles with the axes,
$\therefore$ its slope $= \pm 1$ [Art. 32, Ex. 3]

i.e. $-\frac{b^2 x_1}{a^2 y_1} = \pm 1$ ...(I)

Also $\because$ (x_1, y_1) lies on the ellipse,

$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ ...(2)

(I) Taking +ve sign on the R.H.S. of (I), $-\frac{b^2 x_1}{a^2 y_1} = 1$

or $y_1 = -\frac{b^2 x_1}{a^2}$ ...(3)

Substitute the value of y_1 $\left(= -\frac{b^2 x_1}{a^2}\right)$ from (3) in (2), and find the values of x_1.

Substitute these values of x_1 one by one in (3), and find the corresponding values of y_1.

(II) Taking −ve sign on the R.H.S. of (I),

$-\frac{b^2 x_1}{a^2 y_1} = -1$ or $y_1 = \frac{b^2 x_1}{a^2}$ ...(4)

Proceed again as above.

8. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Draw a rough Fig.

Find the equation of the normal at $\left(-ae, \frac{b^2}{a}\right)$.

If it passes thro' $(0, -b)$, then it will be found that

$ab = a^2 - b^2$ [But $b^2 = a^2(1-e^2)$ or $b = a\sqrt{1-e^2}$]

or $a.a\sqrt{1-e^2} = a^2 - a^2(1-e^2) = a^2 e^2$ [Cancel a^2]

or $\sqrt{1-e^2} = e^2$. Square and transpose.

12. The equations of the ellipses are $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$ ...(1)

$\frac{x^2}{a^2} + \frac{y^2}{a^2+b^2} = 1$ ...(2)

Let m be the slope of a tangent to the ellipse (1).

Then its equation is $y = mx \pm \sqrt{(a^2+b^2)m^2 + b^2}$ ...(3)

$[y = mx \pm \sqrt{a^2 m^2 + b^2}]$

If it touches the ellipse (2), then

$\pm\sqrt{(a^2+b^2)m^2 + b^2} = \pm\sqrt{a^2 m^2 + (a^2+b^2)}$. $[c = \pm\sqrt{a^2 m^2 + b^2}]$

Square, and find the values of m. Substitute these values of m one by one in (3).

15. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

Find the condition that the line $x \cos \alpha + y \sin \alpha = p$ ...(2) may touch the ellipse (1) (Art. 159, Ex. 5). It will be found that

$a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$ or $p = \pm\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$ ...(3)

Taking +ve sign on the R.H.S. of (3), $p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$.

Substituting this value of p in (2), the equation of the tangent is

$x \cos \alpha + y \sin \alpha = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$ ...(4)

Changing α to $90° + \alpha$, the equation of a $\perp$ tangent is

$-x \sin \alpha + y \cos \alpha = \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$ ...(5)

Eliminate α from (4) and (5) [by squaring (4) and (5), and adding].

18. Let (x_1, y_1) be the pole of *any* line w.r.t. the ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

Then the equation of the line (*i.e.* polar) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(1)

If it touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

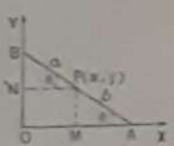
[Compare (1) with $lx+my+n=0$,
here $l=\frac{x_1}{a^2}$, $m=\frac{y_1}{b^2}$, $n=-1$]

then $a^2\frac{x_1^2}{a^4}+b^2\frac{y_1^2}{b^4}=1$ [$a^2l^2+b^2m^2=n^2$ (Art. 159, Ex. 6)]

$\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y)]

$$\frac{a^2x^2}{a^4}+\frac{b^2y^2}{b^4}=1.$$

19. Let the line AB slide with its ends on the two $\perp$ lines OA, OB.



Take OA, OB as the axes [Art. 10(*a*), Rule I].

Let P(*x*, *y*) be any pt. on AB, BP=*a*, PA=*b*, $\angle$BAO=θ.

From P draw PM and PN $\perp$s on the *x*-axis and *y*-axis respectively.

Then $\angle$BPN=$\angle$BAO (Corresponding $\angle$s)
$=\theta$.

Now $x=\text{OM}=\text{NP}$ [But $\frac{\text{NP}}{\text{BP}}=\cos\theta$, $\therefore$ NP=BP $\cos\theta$]
$=a\cos\theta$.

Again $y=\text{MP}$ [But $\frac{\text{MP}}{\text{PA}}=\sin\theta$, $\therefore$ MP=PA $\sin\theta$]
$=b\sin\theta$.

$\therefore \cos\theta=\frac{x}{a}$ ...(1), $\sin\theta=\frac{y}{b}$ ...(2)

Eliminating θ from (1) and (2) [by squaring (1) and (2), and adding],

$1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ [$\because \cos^2\theta+\sin^2\theta=1$]

$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the required locus. This is an ellipse.

21. Proceed as in Ex. 19. It will be found that the equation of the ellipse is $\frac{x^2}{(16)^2}+\frac{y^2}{(4)^2}=1$.

Here (greater denominator) $a^2=(16)^2$,
(lesser denominator) $b^2=(4)^2$.

Now $b^2=a^2(1-e^2)$, $\therefore (4)^2=(16)^2(1-e^2)$. Find the value of e.

22. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Take the equation of the tangent at $(a\cos\theta, b\sin\theta)$.

23. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let P be the pt. $(a\cos\theta, b\sin\theta)$. It will be found that T is $\left(\frac{a}{\cos\theta}, 0\right)$, and N is $(a\cos\theta, 0)$.

$\therefore$ the equation of the circle on NT as diameter is
$(x-a\cos\theta)(x-a\sec\theta)+(y-0)(y-0)=0$
or $x^2+y^2-ax(\cos\theta+\sec\theta)+a^2=0$.

The equation of the auxiliary circle is
$x^2+y^2-a^2=0$.

Show that these two circles cut orthogonally. (Use the condition of orthogonality : $2g_1g_2+2f_1f_2=c_1+c_2$.)

24. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

It will be found that the equation of the tangent at the pt. 'α' is

$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1$ ...(1)

The equation of the auxiliary circle is
$x^2+y^2=a^2$ ...(2)

The equation of the lines joining the origin to the pts. of intersection of the tangent (1) and the auxiliary circle (2) is

$$x^2+y^2=a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2.$$

If these lines are $\perp$, then ?

[Transpose, and use coeff. of x^2+coeff. of $y^2=0$.]

Use $b^2=a^2(1-e^2)$.

25. Proceed as in solved Ex. 2, Art. 169.

26. The equation of the line is $x\cos\alpha+y\sin\alpha=p$ ...(1)

and that of the ellipse is $\frac{x^2}{\frac{1}{2}}+\frac{y^2}{\frac{1}{3}}=1$.

[Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$]

Here $a^2=\frac{1}{2}$, $b^2=\frac{1}{3}$, $\therefore a=\frac{1}{\sqrt{2}}$, $b=\frac{1}{\sqrt{3}}$.

$\therefore$ the equation of the normal at $(a\cos\theta, b\sin\theta)$, namely

$\frac{a^2x}{a\cos\theta}-\frac{b^2y}{b\sin\theta}=a^2-b^2$, *i.e.* $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ becomes

$$\frac{x}{\sqrt{2}\cos\theta}-\frac{y}{\sqrt{3}\sin\theta}=\tfrac{1}{2}-\tfrac{1}{3}=\tfrac{1}{6}.$$

Proceed as in solved Ex. 2, Art. 169.

27. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let P be the pt. $(a\cos\theta, b\sin\theta)$.

Find the equation of the normal at P, and its slope. Find also the slope of CP.

It will be found that if ϕ is the angle between CP and the normal at P,

then $\tan\phi=\frac{a^2-b^2}{ab}\sin\theta\cos\theta=\frac{a^2-b^2}{2ab}\sin 2\theta$.

Now $\tan\phi$ is maximum when $\sin 2\theta$ is maximum, *i.e.* $=1$.

28. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Let P be the pt. $(a\cos\theta, b\sin\theta)$ on the ellipse.

Then the corresponding pt. Q on the auxiliary circle is $(a\cos\theta, a\sin\theta)$. [Art. 169, Cor. 2]

The equation of the auxiliary circle is $x^2+y^2=a^2$ ...(2)

It will be found that the equation of the normal at P to the ellipse (1) is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ...(3)

and that of the normal at Q to the auxiliary circle (2) is

$y=x\tan\theta$ ...(4)

To find the locus of the pt. of intersection of the normals (3) and (4), we have to eliminate θ from (3) and (4).

From (4), $\tan\theta=\frac{y}{x}$, find the values of $\cos\theta$ and $\sin\theta$, and substitute these values in (3).

29. (*a*) The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Take the equation of the tangent at $(a\cos\theta, b\sin\theta)$.

(*b*) Take the equation of the normal at P$(a\cos\theta, b\sin\theta)$.

It will be found that the normal meets the x-axis where, putting $y=0$ in the equation of the normal, $x=\frac{a^2-b^2}{a}\cos\theta$.

$\therefore$ G is $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$, *i.e.* putting $b^2=a^2(1-e^2)$,

G is $(ae^2\cos\theta, 0)$.

30. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the two pts. on the major axis be each distant d from the centre. Then the chord $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$ passes thro' $(d, 0)$, and the chord $\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}=\cos\frac{\gamma-\delta}{2}$ passes thro' $(-d, 0)$.

$$\therefore\quad \frac{d}{a}\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$$

and $$-\frac{d}{a}\cos\frac{\gamma+\delta}{2}=\cos\frac{\gamma-\delta}{2}\quad\text{[To eliminate } d\text{]}$$

Dividing,

$$\frac{\cos\frac{\alpha+\beta}{2}}{-\cos\frac{\gamma+\delta}{2}}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\gamma-\delta}{2}}$$

or $$\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}=\frac{-\cos\frac{\gamma+\delta}{2}}{\cos\frac{\gamma-\delta}{2}}.$$

Apply componendo and dividendo.

31. The equation of the circle is $x^2+y^2=a^2$ ...(1)

and that of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(2)

Any pt. on the circle (1) is P$(a\cos\theta, a\sin\theta)$.

It will be found that the equation of the chord of contact of tangents from P to the ellipse (2) is

$$\frac{x\cos\theta}{a}+\frac{ay\sin\theta}{b^2}=1\quad ...(3)$$

Let (x_1, y_1) be the mid-pt. of *any* chord of the ellipse (2).

It will be found that the equation of the chord is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\quad ...(4)$$

If it is the same as the equation of the chord of contact (3), then comparing coeffs. in (3) and (4), it will be found that

$$\frac{a\cos\theta}{x_1}=\frac{a\sin\theta}{y_1}=\frac{1}{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}$$

$\therefore\ \cos\theta=?$ ...(5) $\sin\theta=?$ ...(6)

Eliminate θ from (5) and (6) [by squaring (5) and (6), and adding].

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

32. It will be found that the foci are $(\pm 8, 0)$. Find the equations and lengths of the joins of
(i) $(8, 0)$ and $(8,$ [illegible]$)$; (ii) $(-8, 0)$ and $(8,$ [illegible]$)$.

33. In the Fig. of Art. 172(e),

$$AG \cdot GA' = (CA + CG)(CA' - CG)$$
$$= (a + e^2x_1)(a - e^2x_1)$$
$$[\because CG = e^2 . x_1 \text{ (Art. 172(d), N.B.)}]$$
$$= a^2 - e^4x_1^2.$$

34. The equations of the two st. lines are $\frac{tx}{a} - \frac{y}{b} + t = 0$ ...(1)

$$\frac{x}{a} + \frac{ty}{b} - 1 = 0 \quad ...(2)$$

To find the locus of the pt. of intersection of the lines (1) and (2), we have to eliminate the variable parameter t from (1) and (2).
[Rule of Note after solved Ex., Art. 75]

35. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let (x_1, y_1) be the pole of the chord. Then the equation of the chord (i.e. polar) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

$$\therefore \text{ the slope of the chord} = -\frac{\frac{x_1}{a^2}}{\frac{y_1}{b^2}} = -\frac{b^2x_1}{a^2y_1}.$$

$\therefore$ the equation of the diameter which bisects the chord is

$$y = -\frac{b^2}{a^2\left(-\frac{b^2x_1}{a^2y_1}\right)}x \qquad [y = -\frac{b^2}{a^2m}x \text{ (Art. 172 (g), N.B.)}]$$

or $y = \frac{y_1}{x_1}x$. Show that the pole (x_1, y_1) lies on this diameter.

36. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the equation of the diameter (i.e. line thro' the centre) be
$y = mx$ ...(1)
Let (x_1, y_1) be any pt. on it, so that $y_1 = mx_1$ ...(2)
The equation of the polar of (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

$$\therefore \text{ the slope of the polar} = -\frac{\frac{x_1}{a^2}}{\frac{y_1}{b^2}} = -\frac{b^2x_1}{a^2y_1}$$

[Substitute the value of y_1 from (2)]

$$= -\frac{b^2x_1}{a^2mx_1} = -\frac{b^2}{a^2m}.$$

Now the equation of the diameter conjugate to the diameter (1) is

$$y = -\frac{b^2}{a^2m}x$$

$\therefore$ the slope of the polar = the slope of the conjugate diameter
$\therefore$ the polar is ∥ to the conjugate diameter.

CHAPTER IX

Pages 281—283.

6. It will be found that the equation of the curve, i.e. hyperbola is $\frac{y^2}{4} - \frac{x^2}{9} = 1$ ...(1)
[R.H.S. = 1, and coeffs. of x^2, y^2 as denoms. (*Note this step*)]
Here denom. of y^2 is +ve, and that of x^2 is −ve [Second standard form]
$\therefore$ it is a hyperbola with transverse axis *vertical* (Art. 176, Ex. 1).
From (1), (+ve denom.) $a^2 = 4$, (−ve denom.) $-b^2 = -9$, $\therefore b^2 = 9$.
Now $b^2 = a^2(e^2 - 1)$, $\therefore 9 = 4(e^2 - 1)$, $\therefore e = ?$
The foci S, S′ lie on the *transverse* axis, i.e. on the y-axis on opposite sides of C such that $CS = CS' = ae = ?$ $\therefore$ the foci are ?

Page 284. Art. 178.

1. (a) Proceed as in solved Ex., Art. 154.

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5. Proceed as in solved Ex. 2, Art. 88.
7. Let (x_1, y_1) be the pole of *any* line w.r.t. the parabola $y^2 = 4ax$. Then the equation of the line (i.e. polar) is $yy_1 = 2a(x + x_1)$
or $2ax - yy_1 + 2ax_1 = 0$ ...(1) [Compare with $lx + my + n = 0$
here $l = 2a$, $m = -y_1$, $n = 2ax_1$]

If it touches the hyperbola $x^2 - y^2 = a^2$ or $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ ...(2)
[Compare with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, here $b^2 = a^2$]
then $a^2(2a)^2 - a^2(-y_1)^2 = (2ax_1)^2$. $[a^2l^2 - b^2m^2 = n^2$ (Art. 179, Ex. 5)]
Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

Page 290.

1. (b) Let (x_1, y_1) be the pole of *any* line w.r.t. the hyperbola $x^2 - y^2 = a^2$. Then the equation of the line (i.e. polar) is

$$xx_1 - yy_1 = a^2.$$

If it touches the circle $x^2 + y^2 = a^2$, then ? (Use the tangent property.) Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

Page 292.

1. Proceed as in solved Ex., Art. 103.

3. The equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let P be the pt. $(a \sec\theta, b\tan\theta)$. Then the co-ordinates of the mid-pt. of AP are $x=\frac{a+a\sec\theta}{2}$, $y=\frac{0+b\tan\theta}{2}$

or $$x=\frac{a(1+\sec\theta)}{2},\ y=\frac{b\tan\theta}{2}.$$

$$\therefore \sec\theta=\frac{2x-a}{a} \quad ...(1),\quad \tan\theta=\frac{2y}{b} \quad ...(2)$$

Eliminate θ from (1) and (2) [by squaring (1) and (2), and subtracting, and using $\sec^2\theta-\tan^2\theta=1$].

Page 293, Art. 185 (*a*).

Ex. Proceed as in solved Ex. 2, Art. 172 (*a*).

Page 294, Art. 185 (*b*).

1. Proceed as in solved Ex., Art. 172 (*e*).

MISCELLANEOUS EXAMPLES ON CHAPTER IX

Pages 294—296.

3. Let the transverse axis $A'A=2a$. $CA=\frac{1}{2}CS$, $\therefore a=\frac{1}{2}ae$, $\therefore e=2$.

4. (*b*) Find Z, the foot of the directrix, *i.e.* the pt. of intersection of the directrix and the line $\perp$ to the directrix, and passing thro' the focus. It will be found that Z is $(-1, 6)$.

The vertices A, A′ divide SZ internally and externally in the ratio $e:1$, *i.e.* in the ratio $2:1$. $\therefore$ A is ? A′ is ? $\therefore$ the centre C, the mid-pt. of AA′, is ?

Let Z′, the foot of the other directrix, be (x, y). Then C is the mid-pt. of ZZ′, $\therefore$?

It will be found that Z′ is $(-3, 8)$.

The other directrix is the line $\parallel$ to the given directrix and passing thro' Z′.

5. Proceed as in solved Ex., Art. 154.

6. The equation of the line is $3x+4y=7$ ...(1)

and that of the hyperbola is $4x^2-3y^2=1$.

Let (x_1, y_1) be the required co-ordinates of the pt.

Find the equation of the tangent to the hyperbola at (x_1, y_1), and hence the equation of the normal at (x_1, y_1).

$\because$ it is the same as the equation of the given line (1),

$\therefore$ compare coeffs., and find x_1, y_1.

7. The equation of the line is $lx+my=1$ ...(1)

and that of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(2)

Substitute the value of $y\left(=\frac{1-lx}{m}\right)$ from (1) in (2). It will be found that

$$x^2(b^2m^2-a^2l^2)+2la^2x-a^2(1+b^2m^2)=0 \quad ...(3)$$

which is a quadratic in x. Let x_1, x_2 be the roots, the abscissae of the pts. of intersection P, Q.

The x-coordinate of R, the mid-pt. of PQ, is

$$x=\frac{x_1+x_2}{2}=\tfrac{1}{2}\,[\text{sum of the roots of the quadratic (3)}]$$
$$=\,?$$

$\because$ R lies on the line (1), $\therefore$ substitute the value of x (found above) in (1), and find the value of y.

9. The equations of the hyperbolas are $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)

and $$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

or $$\frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)}=1 \quad ...(2)$$

$\left(Form:\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right)$ [*Note this step*]

Let m be the slope of a tangent to the hyperbola (1).

Then its equation is $y=mx\pm\sqrt{a^2m^2-b^2}$ ...(3)

If it touches the hyperbola (2), then

$$\pm\sqrt{a^2m^2-b^2}=\pm\sqrt{(-b^2)m^2-(-a^2)}.$$

$[c=\pm\sqrt{a^2m^2-b^2}]$

Square, and find the values of m. Substitute these values of m one by one in (3).

10. Reject the imaginary values of m.

13. *Any* pt. on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(a\cos\theta, b\sin\theta)$.

15. Let (x_1, y_1) be the pole of *any* line w.r.t. the parabola $y^2=4ax$. Then the equation of the line (*i.e.* polar) is $yy_1=2a(x+x_1)$.

If it touches the circle $x^2+y^2=a^2$,

the length of the $\perp$ from the centre $(0, 0)$ on it $=$ the radius

$$\therefore \pm\frac{2ax_1}{\sqrt{4a^2+y_1^2}}=a.$$

Cancel a. Square. $\therefore$ (x_1, y_1) lies on ?

18. The equation of the hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ...(1)

Let P be the pt. $(a \sec\theta, b\tan\theta)$.

19. The equations of the lines are

$$x-y=0 \quad ...(1)$$
$$x+y=0 \quad ...(2)$$

and that of the hyperbola is $x^2-y^2=a^2$.

The equation of the tangent to the hyperbola at *any* pt. $(a\sec\theta, a\tan\theta)$ is $x(a\sec\theta)-y(a\tan\theta)=a^2$

or $$x\sec\theta-y\tan\theta=a \quad ...(3)$$

Find the area of the triangle formed by the lines (1), (2), (3).

20. The equation of the hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

The equation of the normal to the hyperbola at $(a\sec\theta, b\tan\theta)$

is $$\frac{a^2x}{a\sec\theta}+\frac{b^2y}{b\tan\theta}=a^2+b^2 \qquad \left[\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2\right]$$

or $$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2 \quad ...(1)$$

It meets the x-axis where, putting $y=0$ in (1), $x=\dfrac{a^2+b^2}{a}\sec\theta$.

$\therefore$ CM$=\dfrac{a^2+b^2}{a}\sec\theta$, where C is the centre of the hyperbola.

It meets the y-axis where, putting $x=0$ in (1), $y=\dfrac{a^2+b^2}{b}\tan\theta$.

$\therefore$ CN$=\dfrac{a^2+b^2}{b}\tan\theta$.

$\therefore$ the co-ordinates of P are

$$x=\text{CM}=\frac{a^2+b^2}{a}\sec\theta,\ y=\text{MP}=\text{CN}=\frac{a^2+b^2}{b}\tan\theta$$

or $$\sec\theta=\frac{ax}{a^2+b^2} \ ...(2),\ \tan\theta=\frac{by}{a^2+b^2} \ ...(3)$$

Eliminate θ from (2) and (3) [by squaring (2) and (3), and subtracting, and using $\sec^2\theta-\tan^2\theta=1$].

21. (*i*) Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ...(1)

and that of the hyperbola be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ...(2)

The equation of the polar of any pt. $(a\cos\theta, b\sin\theta)$ on the ellipse (1) w.r.t. the hyperbola (2) is

$$\frac{x(a\cos\theta)}{a^2}-\frac{y(b\sin\theta)}{b^2}=1 \text{ or } \frac{x\cos\theta}{a}-\frac{y\sin\theta}{b}-1=0.$$

[Compare with $lx+my+n=0$,

here $l=\dfrac{\cos\theta}{a}$, $m=-\dfrac{\sin\theta}{b}$, $n=-1$]

It touches the ellipse (1) if

$$a^2\left(\frac{\cos\theta}{a}\right)^2+b^2\left(-\frac{\sin\theta}{b}\right)^2=(-1)^2 \qquad [a^2l^2+b^2m^2=n^2]$$

or if $\cos^2\theta+\sin^2\theta=1$, which is true.

(*ii*) The equation of the polar of any pt. $(a\sec\theta, b\tan\theta)$ on the hyperbola (2) w.r.t. the ellipse (1) is ? Proceed as in Part (*i*).

22. The equation of the hyperbola is $x^2-y^2=a^2$.

Let (x_1, y_1) be the mid-pt. of *any* chord. It will be found that the equation of the chord is

$$xx_1-yy_1=x_1^2-y_1^2$$

or $$xx_1-yy_1-(x_1^2-y_1^2)=0.$$

[Compare with $lx+my+n=0$,
here $l=x_1$, $m=-y_1$, $n=-(x_1^2-y_1^2)$]

If it touches the parabola $y^2=4ax$, then

$x_1[-(x_1^2-y_1^2)]=a(-y_1)^2$. [$l.n=am^2$ {Art. 135, Ex. 10}]

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

23. The equation of the rectangular hyperbola is $x^2-y^2=a^2$ ...(1)

Let (x_1, y_1) be the mid-pt. of *any* chord. It will be found that the equation of the chord is

$$xx_1-yy_1=x_1^2-y_1^2 \quad ...(2)$$

The equation of the normal to the hyperbola (1) at $(a\sec\theta, a\tan\theta)$ is

$$\frac{a^2x}{a\sec\theta}+\frac{a^2y}{a\tan\theta}=a^2+a^2 \qquad \left[\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2\right]$$

or $$\frac{x}{\sec\theta}+\frac{y}{\tan\theta}=2a \quad ...(3)$$

If the chord (2) is the same as the normal (3), then comparing coeffs. in (2) and (3),

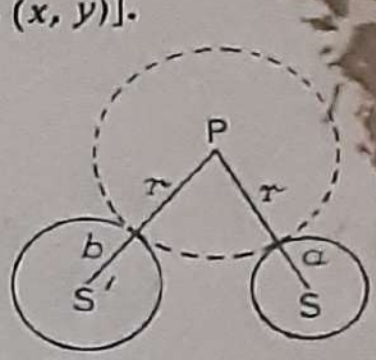
$$\frac{x_1}{\dfrac{1}{\sec\theta}}=\frac{-y_1}{\dfrac{1}{\tan\theta}}=\frac{x_1^2-y_1^2}{2a}$$

or $$\sec\theta=\frac{x_1^2-y_1^2}{2ax_1} \ ...(4),\ \tan\theta=-\frac{x_1^2-y_1^2}{2ay_1} \ ...(5)$$

Eliminate θ from (4) and (5) [by squaring (4) and (5), and subtracting, and using $\sec^2\theta-\tan^2\theta=1$].

Find the locus of (x_1, y_1) [by changing (x_1, y_1) to (x, y)].

24. Let S, S′ be the centres, and a, b the radii of the two given circles.



Let P be the centre, and r the radius of the (variable) circle touching them externally.

Then $S'P - SP = b + r - (a + r)$

$= b - a$, which is constant

[$\because$ it is independent of r]

$\therefore$ the locus of P is a hyperbola whose foci are S, S′

[Art. 185 (a)].